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THESIS

INDEPENDENCE NUMBER OF SPECIFIED I-GRAPHS

by

Matthew S. Dods

June 2020

Co-Advisors:	Thor Martinsen Ralucca Gera Pantelimon Stanica

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INDEPENDENCE NUMBER OF SPECIFIED I-GRAPHS

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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

In this paper, we study the independence number of various classes of I-graphs. I-graphs are generalizations of the Generalized Petersen Graphs. We provide constructions of independent sets given any parameters, as well as bounds for some subclasses of I-graphs. We also prove exact results for some specific I-graphs.

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Table of Contents

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List of Figures

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CHAPTER 1:
Introduction

Graph theory is a field of mathematical study dating back to the eighteenth century. Renowned mathematician Leonhard Euler is credited with the first publication, in which he characterized and solved the historical Königsberg Bridge problem. Since then, graph theory has quickly evolved and established itself as an amazingly useful tool for modeling.

The goal of our research is to further develop our understanding of graph theory by studying the independence number of some classes of I-graphs.

1.1 I-graph Development

Our research subject, I-graphs, has its origins in a famous and widely used graph known as the Petersen Graph. Because of its unique structure, graph theorists often use the Petersen Graph to test or disprove conjectures, making it valuable to understand. Though it has only ten vertices, the Petersen Graph has many properties and symmetries. For example, it is vertex transitive; that is, assuming the vertices are unlabeled, it is impossible to distinguish between any two vertices. The same applies for the edges of the graph. The graph is now very well understood, but its generalizations have not been fully characterized.

A first such generalization is aptly named a *Generalized Petersen Graph*. These graphs can be chosen to be any size, and can have slightly different constructions. Despite these changes, the Generalized Petersen Graphs retain many of the original properties. At this point, we have transitioned from a single graph to countably infinitely many graphs that we call a class of graphs. This vast space of possible Generalized Petersen Graphs has been the subject of extensive research, and much about this class has been understood.

The next level generalization creates the I-graphs. As the Petersen Graph is an element of the Generalized Petersen Graphs, the Generalized Petersen Graphs are a subset of the I-graphs. With yet another parameter added to modify the construction, the class of I-graphs is more complex and larger. While researchers have begun to unearth the properties of these graphs, there is clearly much more to be realized. It is with this class of graphs that our research lies.

1.2 Relevance

Graph theory has been in the forefront of modeling discrete systems for years. Graphs present a reach structure, since the edge set captures the relationships between the objects to be studied.

Among the most apparent and direct applications are cyber networks. When cyber systems are linked together in any form, it can be modeled as vertices with edges between them. Such models, armed with the tools of graph theory, can help understand things like traffic flow, points of failure, central systems, and vulnerabilities. Social networks, too, can be modeled and analyzed as graphs to suggest certain group dynamics based on patterns of interaction.

The research we present makes contributions to understanding a few very specific classes of graphs. Any advancement of what is known within the field adds possible tools to be used to understand models such as those above. The results we present rely on intrinsic structural characteristics of the I-graphs chosen, and identifying and understanding these types of characteristics is necessary for continuing research.

In Chapter 2, we will be presenting many of the existing results regarding Generalized Petersen Graphs and I-graphs, before presenting novel results in Chapter 3 and summarizing them in Chapter 4.

CHAPTER 2: Background

In this chapter, we explore prior research to Generalized Petersen Graphs and I-Graphs. This chapter is organized into sections including Graph Theory Overview, Generalized Petersen Graphs, and I-Graphs.

2.1 Graph Theory Overview

Graph theory is a constantly developing tool in discrete mathematics for visualizing and proving results on relations on a set. Graph theory has been applied to countless areas, from protein composition [1], to social networks, to delivery routes, to cyber networks, and many more. Graphs can be weighted or unweighted and directed or undirected. When a graph is unweighted and undirected, we describe it as "simple." This is the only type of graph we will be considering in this thesis. Before we go further, we will recall the definition a graph G as:

Definition 1. *Graph*

A graph G is a pair of sets (V, E) such that $E \subset V \times V$ of unordered pairs of V. We call the set $V = V(G)$ the set of vertices and the set $E = E(G)$ the set of edges. An edge $\{x, y\}$ joining the vertices x and y is denoted xy [2].

Throughout, we will be using the terms "nodes" and "vertices" interchangeably.

As defined above, graphs describe relations on sets. These relations are shown in the edge set E.

Definition 2. *Adjacency*

In a graph G, two vertices $u, v \in V$ are said to be ***adjacent*** *if $uv \in E$. Subsequently, vertices $u, v \in V$ as above are* ***nonadjacent*** *if $uv \notin E$*

In this paper, however, we will be more interested in non-adjacent vertices within a graph.

Definition 3. *Independent Set*

A subset of vertices such that each are pairwise non-adjacent is called an ***independent set****, denoted* $I(G)$*.*

When analyzing graphs, it is often more important to know the size of an independent set, without knowing each element.

Definition 4. *Independence Number*

The ***independence number****,* $\alpha(G)$*, is the cardinality of a largest independent set.*

It is clear that such a set is not unique in general, though the independence number is. We will be exploring these concepts later when applied to specific classes of graphs.

We now also recall the definitions of graph isomorphism [3], graph automorphism [4], vertex transitive graphs [4], and edge transitive graphs [4].

Definition 5. *Graph Isomorphism*

An ***isomorphism*** *from a graph* G *to a graph* H *is a bijection* $\phi : V(G) \to V(H)$ *such that* $\phi(u)\phi(v) \in E(H)$ *if and only if* $uv \in E(G)$*. We say "*G *is* ***isomorphic*** *to* H*", written* $G \cong H$ *if there is an isomorphism from* G *to* H*.*

Understanding graph isomorphisms, we can build a set of functions describing them.

Definition 6. *Graph Automorphism*

A ***graph automorphism*** *on a graph* G *is an isomorphism mapping* G *to itself. The set of all automorphisms of* G *is called the* ***automorphism group*** *of* G*, denoted* $Aut(G)$*, which satisfies group properties under composition.*

Using automorphisms, we can rigorously describe some symmetry properties of graphs.

Definition 7. *Vertex Transitive*

A graph $G = (V, E)$ *is called* ***vertex transitive*** *if for any two vertices* $u, v \in V$*,* $\exists \phi \in Aut(G)$ *such that* $\phi(u) = v$*.*

Vertex transitivity acts on any pair of vertices in a graph, but if we want to see edges preserved, we must look at edge transitivity.

Definition 8. *Edge Transitive*

A graph $G = (V, E)$ *is called* ***edge transitive*** *if for any two edges* $u_1v_1, u_2v_2 \in E$, $\exists \phi \in Aut(G)$ *such that* $\phi(u_1)\phi(v_1) = u_2v_2$.

With these definitions, we can fully review existing results on some classes of graphs, specifically the Generalized Petersen Graphs and the I-Graphs.

2.2 Generalized Petersen Graphs

The Petersen Graph is a commonly used graph which gained renown following a publication by Julius Petersen in 1898 [4]. The Petersen Graph, Figure 2.1, was constructed as a counterexample to the conjecture that all connected, bridgeless, 3-regular graphs have an edge-coloring of three colors [4].

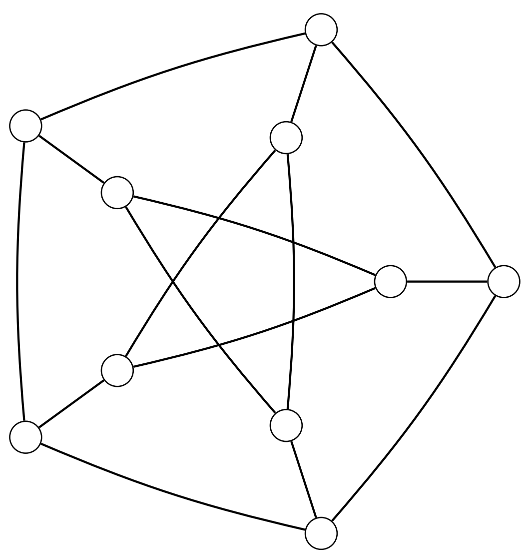

Figure 2.1. The Petersen Graph

In 1950, Harold Coxeter introduced a family of graphs that would later be known as the *Generalized Petersen Graphs* [5].

Definition 9. *Generalized Petersen Graph*

The Generlized Petersen Graph, $P(n,k)$, has vertices and edges given by

$$V(P(n,k)) = \{a_i,\ b_i \mid 0 \le i \le n-1\},$$

$$E(P(n,k)) = \{a_i a_{i+1},\ a_i b_i,\ b_i b_{i+k} \mid 0 \le i \le n-1\},$$

where the subscripts are expressed as integers modulo n. Define $A(n,k)$ (respectively, $B(n,k)$) to be the subgraph of $P(n,k)$ on vertices $\{a_i \mid 0 \le i \le n-1\}$ (respectively, $\{b_i \mid 0 \le i \le n-1\}$) with edges $\{a_i a_{i+1} \mid 0 \le i \le n-1\}$ (respectively, $\{b_i b_{i+k} \mid 0 \le i \le n-1\}$). We will call $A(n,k)$ the outer subgraph and $B(n,k)$ the inner subgraph.

Note: it is clear that $P(n,k) \cong P(n,n-k)$, so we will consider $k \le \lfloor n/2 \rfloor$.

We can think of the Petersen Graph as a set of two rings of five vertices, an inner ring and an outer ring. There are, of course, different depictions which are isomorphic. However, we will use the above picture to foster intuition. Each vertex in the outer subgraph is adjacent to its immediate neighbors in the outer ring and to the corresponding vertex in the inner subgraph. Subsequently, each vertex in the inner subgraph is connected to every second vertex (this can be referred to as a "skip" of 2. Then we can construct the Petersen Graph as the Generalized Petersen Graph $P(5,2)$.

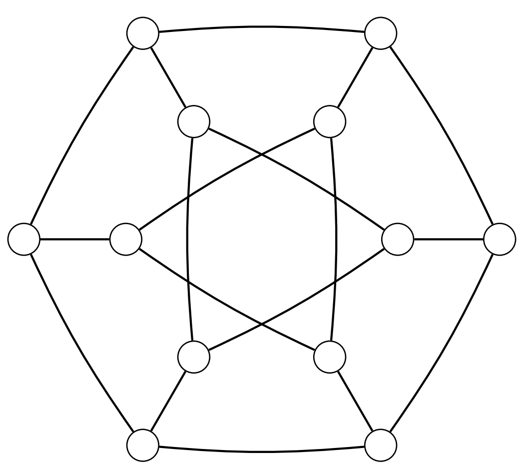

Figure 2.2. $P(6,2)$

Theorem 1 (Fox, Gera, and Stănică [6]). *The following are true for the Generalized Petersen Graphs $P(n,k)$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Note that we take the "skip" $k \leq \lfloor \frac{n-1}{2} \rfloor$, because of the obvious isomorphism $P(n,k) \cong P(n,n-k)$:*

1. *$P(n,k)$ is a 3-regular graph with $2n$ vertices and $3n$ edges.*
2. *$P(n,k)$ is bipartite if and only if n is even and k is odd.*
3. *Assume that n,k,s are positive integers satisfying $k \not\equiv \pm s$ (mod n). Then $P(n,k)$ is isomorphic to $P(n,s)$ if and only if $ks \equiv \pm 1$ (mod n).*
4. *$P(n,k)$ is vertex-transitive if and only if $(n,k) = (10,2)$ or $k^2 \equiv \pm 1$ (mod n).*
5. *$P(n,k)$ is edge-transitive only in the following seven cases: $(n,k) = (4,1)$, $(8,3)$, $(10,2)$, $(10,3)$, $(12,5)$, $(24,5)$.*

Further reading on Petersen Graphs can be found in [4].

2.3 I-Graphs

A further generalization of the Petersen Graph exists, known as an $I-graph$. Recall that in a Generalized Petersen Graph, we can specify both the number of vertices and the skip on the inner subgraph. In an I-graph, we have the same parameters, with the additional parameter of identifying the skip in the outer subgraph as well.

Definition 10. *I-Graph*

An I-graph, $I(n,j,k)$, is a graph whose vertices and edges are given by:

$$V(I(n,j,k)) = \{a_i,\ b_i \mid 0 \leq i \leq n-1\},$$

$$E(I(n,j,k)) = \{a_i a_{i+j},\ a_i b_i,\ b_i b_{i+k} \mid 0 \leq i \leq n-1\}.$$

As above, we will call $A(n,j,k)$ and $B(n,j,k)$ our outer and inner subgraphs, respectively. Again, both are defined as the induced subgraphs on $\{a_i \mid 0 \leq i \leq n-1\}$ and $\{b_i \mid 0 \leq i \leq n-1\}$, respectively.

As we explored earlier, we can see that the Petersen Graph can be constructed as the I-graph $I(5,1,2)$. Clearly any Generalized Petersen Graph $P(n,k) \cong I(n,1,k)$.

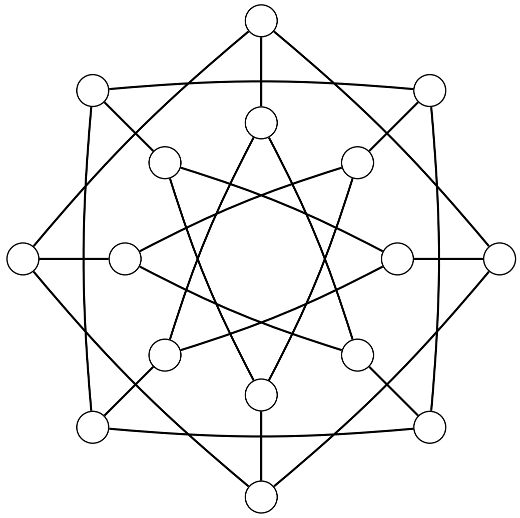

Figure 2.3. $I(8,2,3)$

Boben, Pisanski, and Žitnik give the following properties of I-graphs:

Theorem 2 (Boben, Pisanski, and Žitnik [7])**.** *The following hold:*

1. *The graph $I(n, j, k)$ is connected if and only if $gcd(n, j, k) = 1$. If $gcd(n, j, k) = d > 1$, then the graph $I(n, j, k)$ consists of d copies of $I(n/d, j/d, k/d)$.*
2. *A connected graph $I(n, j, k)$ is bipartite if and only if n is even and j and k are odd.*
3. *If $j \neq \pm k$, then $I(n, j, k)$ has a cycle of lenghth 8. if $j = \pm k$ then $I(n, j, k)$ has a cycle of length 4.*
4. *Let n, j, k and a be positive integers such that $gcd(n, j, k) = 1$ and $gcd(n, a) = 1$. Then the graph $I(n, aj, ak)$ is isomorphic to $I(n, j, k)$.*

Having added an additional parameter to the graph construction, it may seem the graphs have grown greatly in complexity. It turns out that many of the graphs within the class of I-graphs are isomorphic to Generalized Petersen Graphs. $I(8, 2, 3)$, for example, is isomorphic to $P(8, 2)$. Boben et al. give a complete description of such I-graphs through the following theorem and corollary.

Theorem 3 (Boben, Pisanksi, and Žitnik [7])**.** *Let n, j, k be positive integers such that $gcd(n, j, k) = 1$, $gcd(n, j) \neq 1$, and $gcd(n, k) \neq 1$. Then the graph $I(n, j, k)$ is neither vertex transitive nor edge transitive.*

Corollary 3.1 ([7])**.** *A graph $I(n, j, k)$ is a Generalized Petersen graph if and only if $gcd(n, j) = 1$ or $gcd(n, k) = 1$. If $gcd(n, j) = 1$ then $I(n, j, k) \cong P(n, r)$, where r is the solution of the equation $k \equiv r \cdot j \pmod{n}$.*

We call such I-graphs which are connected and are not isomorphic to Generalized Petersen Graphs *proper* I-graphs.In fact, there is such an overlap between Generalized Petersen Graphs and I-graphs that there are no proper I-graphs for $n \leq 11$.

Corollary 3.2. *There are no proper I-graphs with n prime number of vertices.*

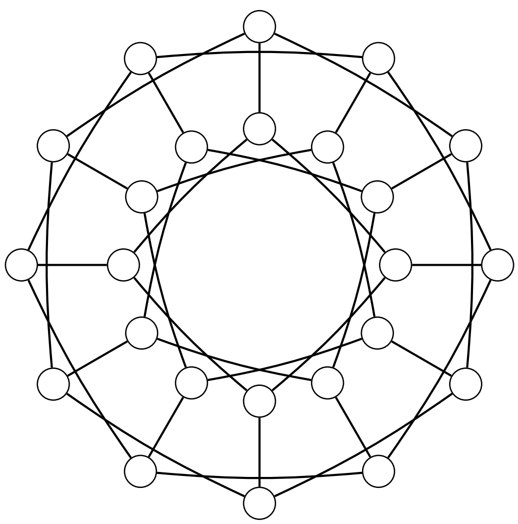

Figure 2.4. $I(12, 2, 3)$

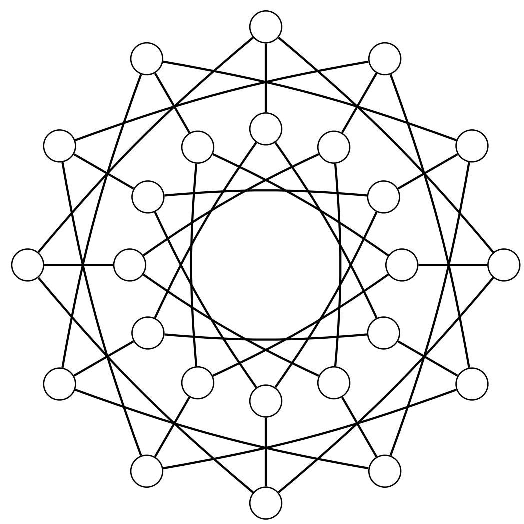

Figure 2.5. $I(12, 3, 4)$

Figure 2.6. $I(12, 2, 3)$ and $I(12, 3, 4)$ are the smallest proper I-graphs.

In Chapter 3, we will present novel results about the independence numbers of some classes of I-graphs.

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CHAPTER 3:
Results

In this chapter, we present various novel results about the independence numbers of I-graphs. We first show a lower bound for a specific class of I-graph via an explicit construction, then move on to show that the lower bound is, for a specific subclass, tight. Further, we present a constructive lower bound for any proper I-graph.

3.1 The Independence Number of $I(12r, 3, 4)$

As discussed in Chapter 2, $I(12, 3, 4)$ is one of the smallest proper I-graphs, alongside $I(12, 2, 3)$. For this reason, we chose to evaluate $I(12r, 3, 4)$, a class of proper I-graphs containing $I(12, 3, 4)$.

Theorem 4. *For proper I-graphs $I(12r, 3, 4)$, $\forall r \in \mathbb{N}$,*

$$\left\lfloor \frac{21r}{2} \right\rfloor \leq \alpha(I(12r, 3, 4)) \leq 12r - 2\delta(r),$$

where $\delta(r) = \begin{cases} 0, & \textit{if } r \textit{ is even} \\ 1, & \textit{if } r \textit{ is odd.} \end{cases}$

Proof. Toward proving the lower bound, consider the following constructions in the three cases:

Case 1: r even. The set

$$A = \left\{ a_{0+8s},\ a_{1+8s},\ a_{2+8s},\ b_{3+8s},\ b_{4+8s},\ b_{5+8s},\ b_{6+8s} \mid s \in \mathbb{N} \cup \{0\},\ s \leq \frac{3r}{2} - 1 \right\}$$

is an independent set. To see this, note that there are no such a_i, $b_j \in A$ such that $i = j$. There are no such a_i, $a_j \in A$ such that $i = j + 3$. There are no such b_i, $b_j \in A$ such that $i = j + 4$. A has $\frac{12r}{8} \cdot 7 = \frac{21r}{2}$ elements. Then $\frac{21r}{2} \leq \alpha(I(12r, 3, 4))$ for r even.

Case 2: $r = 1$. The set

$$A = \{b_0,\ b_1,\ ,a_2,\ a_3,\ a_4,\ b_6,\ b_7,\ a_8,\ a_9,\ a_{10}\}$$

is an independent set, by the same logic as Case 1. See Figure 3.1. Then A has $\lfloor \frac{21r}{2} \rfloor = 10$. Then $\lfloor \frac{21}{2} \rfloor \leq \alpha(I(12,3,4)$.

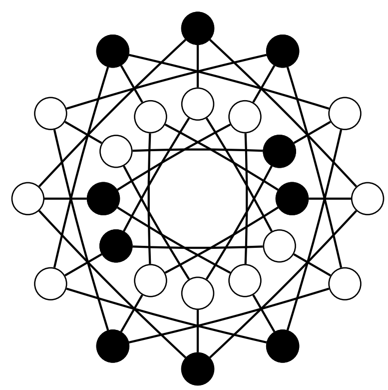

Figure 3.1. Case 2: $\alpha(I(12,3,4)) = 10$

Case 3: $r \geq 3$ **odd.** The set $A = \{\mathrm{b}_0, b_1, , a_2, a_3, a_4, b_6, b_7, a_8, a_9, a_{10}\} \bigcup \{a_{12+8s}, a_{13+8s}, a_{14+8s}, b_{15+8s}, b_{16+8s}, b_{17+8s}, b_{18+8s} \mid s \in \mathbb{N} \cup \{0\}, s \leq \frac{3(r-1)}{2} - 1\}$ is an independent set, by the same logic as Case 1. A has $10 + \frac{12r-12}{8} \cdot 7 = 10 + \frac{21(r-1)}{2} = \lfloor \frac{21r}{2} \rfloor$. Then $\lfloor \frac{21}{2} \rfloor \leq \alpha(I(12r,3,4))$ for r odd.
Since the cases are exhaustive, and since $\lfloor \frac{21r}{2} \rfloor \leq \frac{21}{2}\ \forall r \in \mathbb{N}$, we have that $\lfloor \frac{21r}{2} \rfloor \leq \alpha(I(12r,3,4))$.

Toward proving the upper bound, consider first the cycles of $B(12r,3,4)$ (the inner subgraph). Each cycle is of length $\frac{12r}{4} = 3r$. For r even, since each cycle is of even length, we can take every second node in each cycle, for a total of $\frac{3r}{2}$ nodes. There are four such cycles, so we can take at most $6r$ nodes from $B(12r,3,4)$. For r odd, each cycle is of odd length, so we can take at most $\frac{3r-1}{2}$ nodes from each. There are four such cycles, so we can take at most $6r-2$ nodes from $B(12r,3,4)$.
Consider now the cycles of $A(12r,3,4))$ (the outer loop). Each cycle is of length $\frac{12r}{3} = 4r$. Then for all r, each cycle is of even length. As before, we can take at most $\frac{4r}{2} = 2r$ nodes from each cycle. There are three such cycles, so we can take at most $6r$ nodes from $A(12r,3,4)$.
Thus, we can take at most $12r - 2 \cdot \delta(r)$ from $I(12r,3,4)$, where $\delta(r) = \begin{cases} 0, & \text{if } r \text{ is even} \\ 1, & \text{if } r \text{ is odd} \end{cases}$. $\square$

Remark. *It is important to note that the upper bound is attainable, but not sharp. For* *example, for $I(12,3,4)$, we have that* $\lfloor \frac{21\cdot 1}{2} \rfloor = 10 \leq \alpha(I(12,3,4)) \leq 12\cdot 1 - 2\cdot 1 = 10$.

We have established loose bounds, and acknowledge that the upper bound is not sharp. We now consider the independence number of I-graphs for specific values of $0 < r \in \mathbb{Z}$. Surely, we can do the next theorem computationally, but we wanted to argue that even for some specific small cases, a theoretical proof is quite involved.

Theorem 5. *We have* $\alpha(I(24,3,4)) = 21$.

Proof. By the lower bound presented above, we know that $\alpha(I(24,3,4)) \geq 21$. Suppose there exists an independent set such that $\alpha(I(24,3,4)) = 22$. As we have seen before, each ring can contain at most $\frac{n}{2}$ independent nodes, in this case twelve. Then there are three possible configurations: $A(24,3,4)$ contains twelve nodes of our independent set, each $A(24,3,4)$ and $B(24,3,4)$ contain eleven nodes of our independent set, or $B(24,3,4)$ contains twelve nodes of our independent set. We will handle these each in separate cases.

Case 1: $A(24,3,4)$ **contains twelve nodes of our independent set.** There are exactly two ways to achieve this: groups of three or every other node in the outer ring. Then we must select the inner nodes. In the first pattern, the available nodes on the inner ring come in four blocks of three. To select 10 nodes, by the pigeonhole principle, at least one block must have all three selected. But the boundary nodes are adjacent to nodes in the neighboring groups of three, eliminating them from being used. Then there are 7 spaces left from which to choose 7 nodes, but by the same logic any boundary node eliminates the boundary node of the next group, thus impossible. In the second pattern, since the outer selection pattern aligns with the inner cycles, we can take at most half of the remaining 12 nodes in the inner ring.

Case 2: $B(24,3,4)$ **contains twelve nodes of our independent set.** Again, there are exactly two ways to ways to achieve this: choosing the set $\{b_{8i}, b_{8i+1}, b_{8i+2}, b_{8i+3} | i \in \{0,1,2\}\}$ or the set $\{b_{8i}, b_{8i+2}, b_{8i+3}, b_{8i+5} | i \in \{0,1,2\}\}$. Consider the first pattern. Again, by the pigeonhole principle, since this pattern repeats three times, one off-grouping in $A(24,3,4)$ must contain at least 4 nodes. This is obviously impossible. Consider the second pattern. Again, we must fit 4 nodes in at least one of the groupings, which is impossible. (The most we can take from any is 3).

Case 3: Each $A(24,3,4)$ and $B(24,3,4)$ contain eleven nodes of our independent set. Suppose we chose 11 nodes in the outer ring. Recall that there are three cycles of 8 nodes in the outer ring, and we can take at most four from each. Partition $A(24,3,4)$ into subgraphs A_0, A_1, and A_2 corresponding to each cycle. For example, $A_0 = \{a_0, a_3, a_6, a_9, a_{12}, a_{15}, a_{18}, a_{21}\}$. Then there are two cycles with four nodes selected, and there is one cycle with three selected. Consider now any of the inner cycles. Similarly, we have 4 cycles of 6 nodes each. To achieve eleven nodes, there must be three cycles containing three nodes of our independent set and one cycle containing two nodes of our independent set. Choose any of the inner cycles, and select three nodes. Due to the symmetry of the graph, we can assign the three nodes to labels b_0, b_8, b_{16}. Since $\gcd(8,6) = 2$, we know that each cycle in $B(24,3,4)$ will be adjacent to exactly two nodes in each cycle of $A(24,3,4)$. Consider $A_0 = \{a_0, a_3, a_6, a_9, a_{12}, a_{15}, a_{18}, a_{21}\}$. Then a_0 is adjacent to b_0, and therefore cannot be selected. Since we must have four nodes selected, we must choose a_3, a_9, a_{15}, and a_{21}. Now consider $A_1 = \{a_1, a_4, a_7, a_{10}, a_{13}, a_{16}, a_{19}, a_{22}\}$. Again, since we must have four nodes selected, we must choose a_1, a_7, a_{13}, and a_{19}. Finally, consider $A_2 = \{a_2, a_5, a_8, a_{11}, a_{14}, a_{17}, a_{20}, a_{23}\}$. We need only choose three nodes from A_2. Before we can choose which three nodes, we must consider the next cycle in $B(24,3,4)$, $\{b_1, b_5, b_9, b_{13}, b_{17}, b_{21}\}$. By our forced selections, b_1, b_9, b_{13}, and b_{21} are ineligible to be selected, so we must ensure nodes b_5 and b_{17} can be selected. This necessitates we choose nodes a_2, a_{14}, and a_{20} from the A_2 subgraph. Finally, consider the third cycle of $B(24,3,4)$, $\{b_2, b_6, b_{10}, b_{14}, b_{18}, b_{22}\}$. We can choose exactly two nodes from this group (anything of the form $\{b_6, b_{10}\} \times \{b_{18}, b_{22}\}$). However, to attain the requisite 11 nodes from the inner ring, we needed to be able to select three nodes. Thus this case is also impossible.

Since the cases are exhaustive, we have that $\alpha(I(24,3,4)) = 21$. □

We have investigated the cases for $r = 3$ ($n = 36$) and $r = 4$ ($n = 48$), and, as with $I(12,3,4)$ and $I(24,3,4)$, we have not been able to identify an independent set with more than $\lfloor \frac{21r}{2} \rfloor$ elements.

We offer now a construction-based formula which can provide a lower bound for any proper I-graph. The construction utilizes a greedy algorithm to quickly generate an independent set. Toward building intuition for the formula, we first define k- groupings and j- groupings.

In order to offer the next theorem, we must first introduce a partition on the I-graph $I(24r, 3, 4)$. Define $Q = \bigcup_{\gamma=0}^{3r-1}(Q_\gamma)$ where $Q_\gamma = \{a_{8\gamma}, b_{8\gamma}, a_{8\gamma+1}, b_{8\gamma+1}, \ldots, a_{8\gamma+7}, b_{8\gamma+7}\}$. We can see that by construction that $Q = I(24r, 3, 4)$ and that $Q_i \bigcap Q_j \forall i \neq j$. Then this is a partition of $I(24r, 3, 4)$.

Lemma 1. *If $Q_i \in Q$ contains eight nodes of an independent set, then Q_{i+1} can contain at most seven such nodes.*

Proof. Assume a set Q_d contains eight nodes of an independent set. We know that if we select a node $a_\ell \in A(24r, 3, 4)$, it is impossible to also select either $a_{\ell-3}$ or $a_{\ell+3}$. Similarly, if we select a node $b_\ell \in B(24r, 3, 4)$, it is impossible to select either $b_{\ell-4}$ or $b_{\ell+4}$ (in both cases, indices are given modulo $24r$). Using this, we know that we can select at most three consecutive nodes in $A(24r, 3, 4)$. Repeating this logic, we know we can select at most four consecutive nodes in $B(24r, 3, 4)$.

Since we must select eight nodes from Q_d, we know that there must be an index i in Q_d such that a_i and b_{i+1} are in our independent set. Since we have now chosen at most seven nodes, there must be an index h such that b_h and a_{h+1} are in our independent set.

Looking now at Q_{d+1}, we see that b_{h+4} and $a_{(h+1)+3}$ are both ineligible to be selected. We know that h must be such that this overlap occurs in Q_{d+1} else we would have failed to select eight nodes in Q_d. Due to this overlap, we see that Q_{d+1} contains at most seven nodes of an independent set. □

Lemma 2. *Between any two sets $Q_\mu, Q_\gamma \in Q$, such that each contain eight nodes of an independent set, there is a set $Q_i \in Q$ such that Q_i contains at most six nodes of our independent set.*

Proof. Assume we have two sets, Q_μ and Q_γ which each contain eight nodes of an independent set. Without loss of generality, assume $\mu < \gamma$. Consider the set $Q_{\mu+1}$. From Lemma 1, we see that it can contain at most seven nodes of an independent set. Again, following the logic of Lemma 1, we see that in order to contain seven nodes, there must be an index h in $Q_{\mu+1}$ such that b_h and a_{h+1} are in the independent set. Looking forward to $Q_{\mu+2}$, we see again that b_{h+4} and $a_{(h+1)+3}$ are both ineligible to be selected, and we know this overlap must occur in $Q_{\mu+2}$ for the same reasons. This pattern continues for all further adjacent sets in Q which contain seven nodes of the independent set.

Now consider $Q_{\gamma-1}$. Again, we know that $Q_{\gamma-1}$ can contain at most seven nodes, and to do so, there is an index ω in $Q_{\gamma-1}$ such that a_ω and $b_{\omega+1}$ are in the independent set. Then in $Q_{\gamma-2}$ both $a_{\omega-3}$ and $b_{(\omega+1)-4}$ are ineligible. This pattern will also extend for all adjacent sets in Q which contain seven nodes.

Finally consider a set $Q_i \in Q$ with $\mu < i < \gamma$, and with Q_{i-1} and Q_{i+1} each containing at least seven nodes of an independent set. As shown above, there is are indices τ and ψ in Q_{i-1} and Q_{i+1}, respectively which cause the overlaps in Q_i as described above. However, since we know that τ is an index in Q_{i-1}, the overlap must occur in the first four positions of Q_i. Similarly, since we know ψ is an index in Q_{i+1}, we know the overlap must occur in the last four positions of Q_i. Therefore, the overlaps cannot occur in the same spot in Q_i, so Q_i can contain at most six nodes of an independent set. □

Remark. *A special case of Lemma* 2 *is when we have exactly one* $Q_i \in Q$ *which contains eight elements of an independent set. In this case, we can see we will still encounter the same problem of needing at least one set in* Q *containing at most six nodes of our independent set.*

Theorem 6. *For* $I(24r, 3, 4)$*, we have that* $\alpha(I(24r, 3, 4)) = 21r$.

Proof. Assume we have a graph $I(24r, 3, 4)$ which has an independent set of $21r$ nodes. We know this is possible by the earlier construction.

Consider a partition $Q = \bigcup_{\gamma=0}^{3r-1}(Q_\gamma)$ of the graph into windows of 8 indices: $Q_\gamma = \{a_{8\gamma}, b_{8\gamma}, a_{8\gamma+1}, b_{8\gamma+1}, \ldots, a_{8\gamma+7}, b_{8\gamma+7}\}$. Since we have $21r$ nodes in our independent set and $3r$ partitions, we know that the partitions must contain an average of 7 nodes each. Assume now, towards contradiction, that we add one more node to the independent set, so $\alpha(I(24r, 3, 4)) = 21r + 1$. By the pigeonhole principle, it must be the case that at least one partition contains eight nodes of our independent set. Since we added a node to exactly one partition, the remaining partitions must still contain an average of seven nodes of the independent set. By the above lemma, one of these partitions must contain at most six nodes of the independent set. In order to keep the average of seven nodes per partition, there must be at least one partition containing eight nodes of the independent set. Using the above lemma again, we see another partition must contain at most six nodes of the independent set. Repeating this across all partitions, we arrive at our contradiction. At least one of the following will hold:

1. Two partitions containing eight nodes of the independent set will be adjacent ($\Rightarrow\Leftarrow$).
2. We will have failed to select $21r + 1$ nodes to be in our independent set ($\Rightarrow\Leftarrow$).

Therefore $\alpha(I(24r, 3, 4)) \leq 21r$, and consequently, putting these bounds together, $\alpha(I(24r, 3, 4)) = 21r$. □

3.2 Other Constructive Bounds

Before we continue with our next constructive bound, we first define k-groupings and j-groupings.

Definition 11. *k-Groupings*

We call a sequence of k consecutively indexed nodes (mod n) from $B(n, j, k)$ a k-grouping.

Definition 12. *j-Groupings*

We call a sequence of j consecutively indexed nodes (mod n) from $A(n, j, k)$ a j-grouping.

We used the ideas of k-groupings and j-groupings in our constructive lower bound for $I(12r, 3, 4)$, but here we need to use them more generally. Now we provide a closed form, constructive lower bound for any given proper I-graph.

Theorem 7. *For any proper I-graph $I(n, j, k)$, with $j < k$ and $\frac{n}{k}$ even, we have:*

$$\frac{n}{2k} \cdot (k + j \cdot \left\lfloor \frac{k}{2j} \right\rfloor + \delta_0(\left\lfloor \frac{k}{j} \right\rfloor) \leq \alpha(I(n, j, k)),$$

where $\delta_0(\lfloor \frac{k}{j} \rfloor) = \begin{cases} k \pmod{j}, & \text{if } \left\lfloor \frac{k}{j} \right\rfloor \text{ is even} \\ j, & \text{if } \left\lfloor \frac{k}{j} \right\rfloor \text{ is odd.} \end{cases}$

Proof. Suppose we have a proper I-graph $I(n, j, k)$ with $j < k$. Suppose also that $\frac{n}{k}$ is even. We choose to look at a an interval of size $2k$. Without losing any generality, we will use indices $\{0, 1, \ldots, 2k-1\}$. We first select a k-grouping into our independent set using nodes $\{b_0, b_1, \ldots b_{k-1}\}$. This leaves only choices from the following set: $\{a_k, a_{k+1}, \ldots, a_{2k-1}\}$. From the remaining k nodes in $A(n, j, k)$, we can choose at least $\frac{k}{2j}$ j-groupings: $\{a_k, a_{k+1}, \ldots a_{k+j-1}\} \cup \{a_{k+2j}, a_{k+2j+1}, \ldots a_{k+3j-1}\} \cup \cdots \cup \{a_{k+2(\lfloor \frac{k}{2j} \rfloor - 1)j}, a_{k+2(\lfloor \frac{k}{2j} \rfloor -1)j+1}, \ldots, a_{k+2\lfloor \frac{k}{2j} \rfloor - j - 1}\}$. We now consider the remaining $k - 2j \cdot \lfloor \frac{k}{2j} \rfloor$, nodes in two cases.

Case 1: $k - 2j \cdot \lfloor \frac{k}{2j} \rfloor < j$. Under this assumption, we can see that $\lfloor \frac{k}{j} \rfloor$ is even. In this case, we have taken as many full j-groupings as possible to be in our independent set. However, since each j-grouping above has been selected with the necessary off-grouping, or a set of nodes that cannot be selected, alongside, we know that all the remaining $k - 2j \cdot \lfloor \frac{k}{2j} \rfloor$ nodes can be selected. Notice that under the assumption that we have fewer than j remaining nodes, $k - 2j \cdot \lfloor \frac{k}{2j} \rfloor$ returns the same value as $k \pmod j$.

Case 2: $k - 2j \cdot \lfloor \frac{k}{2j} \rfloor \geq j$. When this is true, we can see that $\lfloor \frac{k}{j} \rfloor$ is odd. In this case, we can select exactly one more j-grouping to be in our independent set. This is true because, as discussed in Case 1, every earlier j-grouping was chosen on an interval to include its necessary off-grouping.

These cases are exhaustive, and complete a constructive lower bound for any interval of length $2k$. However, we must finally prove that each interval, as constructed, behaves independently. That is, no node selected in one interval disallows the selection of a node in another interval that otherwise would have been selected by the proposed construction. To do so, we will look at three consecutive intervals. If the construction holds for both the interval immediately before and the interval immediately after, then by the symmetry of the I-graphs, we know that we can apply the construction to each of the $\frac{n}{2k}$ intervals.

For simplicity of notation, we will choose our reference interval to use indices $\{2k, 2k+1, \ldots, 4k-1\}$. Call this interval Q_0. Then Q_{-1} uses indices $\{0, 1, \ldots, 2k-1\}$ and Q_1 uses indices $\{4k, 4k+1, \ldots, 6k-1\}$. By our construction, we selected nodes $\{b_{2k}, b_{2k+1}, \ldots, b_{3k-1}\}$ into our independent set. Outside Q_0. This prevents us from selecting nodes $\{b_k, , b_{k+1}, \ldots, b_{2k-1}\}$, none of which are used in this construction. While it is hard to specify in general which indices the final selected group in Q_0 uses, we know the largest it can be is j consecutive nodes. This case prevents us from selecting nodes $\{a_{4k}, a_{4k+1}, \ldots, a_{4k+j}\}$. Since we assumed $j < k$, we know that none of these nodes are used in this construction. Then no nodes selected in Q_0 prevent the selection of nodes in Q_{-1} or Q_1, we have that this is a valid construction, and the bound holds. $\square$

This bound was developed in collaboration with Zachary Klein. It prioritizes selecting complete k-groupings and filling in with as many j-groupings as possible. For this reason, we informally call this the "k-grouping bound". In some cases, such as the class $I(24r, 3, 4)$, this bound is actually equality. Additionally, our colleague, Zachary Klein, has shown equality will hold for the class $I(6r, 2, 3)$ (with $r \geq 2$) [8]. This is not true in general, however. We offer one more bound to demonstrate this.

Theorem 8. *For proper I-graphs $I(15r, 3, 5)$, $\forall r \in \mathbb{N}$,*

$$5 \cdot \left\lfloor \frac{5r}{2} \right\rfloor + \delta(r) \leq I(15r, 3, 5) \leq 15r - 4 \cdot \delta(r),$$

where $\delta(r) = \begin{cases} 0, & \textit{if } r \textit{ is even} \\ 1, & \textit{if } r \textit{ is odd} \end{cases}$.

Proof. Toward proving the lower bound, we offer the following constructions for three cases:

Case 1: r even. The set

$$A = \left\{ a_{0+6s},\ a_{1+6s},\ a_{2+6s},\ b_{3+6s},\ b_{5+6s} | s \in \mathbb{N} \cup \{0\}, s \leq \frac{5r}{2} - 1 \right\}$$

is an independent set. Note that there are no such $a_i, b_j \in A$ such that $i = j$. Similarly, there are no such $a_i, a_j \in A$ such that $i = j + 3$. Finally, there are no such $b_i, b_j \in A$ such that $i = j + 5$. Thus, $|A| = \frac{15r}{6} \cdot 5 = \frac{25r}{2}$, and the bound holds.

Case 2: $r = 1$. The set

$$A = \{a_0,\ a_1,\ a_2,\ b_3,\ b_5,\ , a_6,\ a_7,\ a_8,\ b_9,\ b_{11},\ b_{12}\}$$

is an independent set. See Figure 3.2. Then $|A| = 11$, and the bound holds.

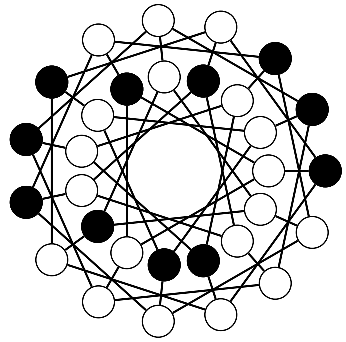

Figure 3.2. Case 2: $\alpha(I(15,3,5)) = 11$

Case 3: r **odd.** The set

$$A = \left\{b_2\} \cup \{a_{3+6s},\ a_{4+6s},\ a_{5+6s},\ b_{6+6s},\ b_{8+6s} | s \in \mathbb{N} \cup \{0\}, s \leq \frac{5r-1}{2} - 1\right\}$$

is an independent set, by the same logic as in Case 1. Then $|A| = 5 \cdot \lfloor \frac{15r}{6} \rfloor + 1$, and the bound holds.

Toward proving the upper bound, first, as we have before, consider the cycles of $B(15r, 3, 5)$. Each cycle is of length $\frac{15r}{5} = 3r$. When r is even, each cycle has even length, so we can take $\frac{3r}{2}$ nodes. When r is odd, we can select $\frac{3r-1}{2}$ nodes. There are five such cycles, so we can take $\frac{15r}{2}$ nodes when r is even or $\frac{15r-5}{2}$ nodes when r is odd.

Consider now the cycles of $A(15r, 3, 5)$. Similarly, each cycle has length $\frac{15r}{3} = 5r$. When r is even, we can take at most $\frac{5r}{2}$ nodes from each. When r is odd, we can take at most $\frac{5r-1}{2}$ nodes from each. There are three such cycles, so we can take $\frac{15r}{2}$ nodes when r is even or $\frac{15r-3}{2}$ nodes when r is odd. Thus we can take at most $15r - 4 \cdot \delta(r)$ from $I(15r, 3, 5)$, where

$$\delta(r) = \begin{cases} 0, & \text{if } r \text{ is even} \\ 1, & \text{if } r \text{ is odd} \end{cases}.$$ □

Corollary 8.1. *We have that* $\alpha(I(15,3,5)) = 11$.

Proof. By applying the bounds given in Theorem 8, we see $11 \leq \alpha(I(15,3,5)) \leq 11$. □

Remark. *Notice that in Theorem 7, we get that* $12r \leq \alpha(I(15r, 3, 5))$*, but we have just* *shown, in Case* 2 *of Theorem* 8*, that when* r *is even, we can select at least* $\frac{25r}{2}$ *nodes. Therefore Theorem* 7 *is not always equality.*

As opposed to Theorem 7, this construction prioritizes selecting complete j-groupings from $B(15r, 3, 5)$, then finding the appropriate nodes from $A(15r, 3, 5)$. This is an example of what we informally call a "j-grouping bound". We do not yet have a closed formula for such a bound, but we conjecture that for any proper I-graph, either a k-grouping bound or a j-grouping bound will be equal to the independence number.

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CHAPTER 4:
Conclusion

In this chapter we will summarize what we present in this thesis in Chapter 2 and Chapter 3. After doing so, we will briefly discuss future work relevant to this thesis.

In this thesis, we investigate the independence number of some classes of I-graphs. We prove bounds for specific classes of I-graphs as well as for a much more general class.

4.1 Summary

In Chapter 2, we offer several definitions, beginning with a simple discrete graph and some other relevant graph definitions. This includes definitions of an independent set and a graph's independence number. We proceed to discuss the Petersen Graph, and the generalizations that took place to get from the Petersen Graph to the Generalized Petersen graphs and finally, to the I-graphs we studied. Of the I-graphs, we chose to study only proper I-graphs. Throughout, we present current results pertaining to each class of graphs.

In Chapter 3, we offer several theorems and detailed proofs of each. We begin by bounding the independence number of the class of graphs of the form $I(12r, 3, 4)$, for $r \in \mathbb{N}$. Next we prove $\alpha(I(24, 3, 4)) = 21$ in order to test our bounds and develop intuition for the general case. We go on to prove a formula which produces the exact independence number for even values of r. From there, we provide a general lower bound for any proper I-graph meeting certain criteria using a method we describe as "k-groupings," which constructs an independent set based on the input parpameters. Finally, we demonstrate that while this method works well in general, it can be improved since we identified a parallel method, which we call "j-groupings", that is better in the class $I(15r, 3, 5)$.

4.2 Future Work

Due to time limitations for this project, and the difficulty of finding the independence number (it is an NP-hard optimization problem, after all), we did not fully characterize the independence numbers of I-graphs. There are several problems directly related to this work that we believe should be studied.

The first is finding and proving equality for the cases of $\alpha\left(I(12r, 3, 4)\right)$ for r odd. We have observed that our presented lower bound seems to be equality, but our k-groupings pattern does not hold throughout.

Next is to identify a general closed-form equation which gives a lower bound for any proper I-graph using a j-grouping construction. This construction requires much greater understanding of the interactions between the parameters j and k than a k-grouping construction.

Finally, we identified two conjectures that we are leaving open for future progress. The first is that for the case of $I\left(r \cdot j(j+1), j, j+1\right)$, with r even, a k-grouping construction will produce a maximum independent set. The second is that for any proper I-graph, either a k-grouping construction or a j-grouping construction will produce a maximum independent set. We have observed both to be true.

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