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NONPARAMETRIC CLASSIFICATION ERROR PROBABILITES FOR TWO DIMENSIONAL EXPONENTIAL POPULATIONS

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NONPARAMETRIC CLASSIFICATION ERROR PROBABILITIES
FOR TWO DIMENSIONAL EXPONENTIAL POPULATIONS

by

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ABSTRACT

An approach to the classification problem is one that is dependent on the amount of information assumed to be known about the distributions of the populations. It is assumed in this thesis that nothing is known about the distributions for a two population case. The probability of misclassification of an individual Z is presented in general. The approach is carried further to develop explicit forms of the error probability when the two populations are bivariate exponential.

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1. INTRODUCTION

The discrimination problem may in a sense be considered that of multiple classification, i.e., an individual Z is known to belong to just one of j categories or populations and it must be classified into one of these populations on the basis of what is known about Z and the existing populations. The problem lends itself to a statistical approach when available information about Z is in the form of observed values of random variables which have probability distributions for each of the different categories or populations.

Discriminatory analysis in its historical evolvement is documented with an extensive bibliography by J. L. Hodges in [1].

The history of discriminatory analysis may be represented in several broad phases of development. A Pearsonian stage is identified with the introduction and use of his coefficient of racial likeness. This stage is considered to be followed by a Fisherian stage associated with the introduction of the linear discriminant function and this stage is followed by a Neyman-Pearson stage and a contemporary Waldian stage. The latter two reflect the introduction of the concepts of probability of misclassification and that of risk into the realm of discriminatory analysis.

For simplicity and ease of computation the discrimination problem will be considered only in the two population case, i.e., the individual Z is known to be distributed over some space according to distribution F or according to distribution G and it is desired to decide which of the distributions Z has on the basis of the observed value z.

An approach to this classification problem is one that is dependent on the amount of information assumed to be known about F and G. This approach allows the problem to be segmented into three types:

(1) F and G are completely known -- On the basis of an observation of Z, the problem is to determine which is the distribution of Z. Treatment of this problem has been extensive and its solution lies within the Neyman-Pearson lemma.

(2) F and G are known but complete knowledge is lacking in its parameters -- F and G are of the same family of distributions but differ parametrically and on the basis of an observation on Z, the problem is to determine which is the distribution of Z. Hodges and Fix discuss this in [2] and identify the most familiar example of this process as the linear discriminant function where the assumption is made that F and G are p-variate normal distributions having the same (unknown) covariance matrix. It is noted that the approach is reasonable if the assumptions are well founded but validity is questionable if the populations are obviously not normal or if they are normal but with obviously unequal covariance matrices.

(3) F and G are completely unknown -- Nothing is assumed about F and G other than their existence and on the basis of an

observation on Z, the problem is to decide which of F and G is the distribution of Z.

The last problem type is a problem of nonparametric classification and is the area of concern of this thesis. The area of nonparametric classification has its possibly first published treatment in Hodges and Fix's [2] and [3]. In these papers Hodges and Fix considered the two population problem, however, they noted that the approach in general, has optimum properties for large samples and applies to cases where there are more than two populations to be discriminated. In [3] a comparison is made of the nonparametric approach and the linear discriminant approach, assuming both populations to be normal with equal covariance matrices.

Eaton in [4] and Hager in [5] extend the work of Hodges and Fix to one dimensional exponential populations. This paper continues the investigation of exponential populations when the distributions are bivariate exponential.

Section 2 will introduce and summarize the concepts and methodology of [3]. Section 3 will apply these concepts to calculate a probability of misclassification utilizing two different distance functions to the two population problem when both populations are bivariate exponential. Section 4 will present the conclusions and recommendations evolving from the effort set forth in Section 3.

2. PERFORMANCE OF A NONPARAMETRIC DISCRIMINATOR WHEN THE TWO POPULATIONS HAVE NORMAL DISTRIBUTIONS WITH EQUAL COVARIANCE MATRICES

The two population classification problem will be defined in the following manner. Let $X_1, X_2, \cdots, X_n$ be a sample from a p-variate distribution F and let $Y_1, Y_2, \cdots, Y_m$ be a sample from the p-variate distribution G. An observation z is made and it is known to be distributed either as F or as G. The problem being to assign z to one of these two.

The approach utilized for the nonparametric procedure will be through the concept of nearness. A distance function is defined in the p-dimensional sample space which allows a ranking of the combined samples according to their nearness to z. Within this framework, z would be assigned to the F population if most of the nearby observations are X's and assigned to G if most of the nearby observations are Y's. For simplification, the sample sizes are assumed to be equal, i.e., m=n. As an assignment criteria, an odd integer k is selected and z is assigned to that distribution from which came the majority of the k nearest observations. The case studied in this thesis will be when k = 1, the rule of the nearest neighbor.

In [2] it is shown that several of these nonparametric discriminators have asymptotically optimum performance as m and n tend to infinity. By this is meant that probabilities of misclassification,

$$P_1 = P\ (Z \text{ is assigned to } G \mid Z \text{ came from } F)$$

$$P_2 = P\ (Z \text{ is assigned to } F \mid Z \text{ came from } G)$$

will approach the theoretical minimum values obtainable if F and G were completely known as the sample size (m,n) tends to infinity.

In [3] Hodges and Fix investigate the probabilities of misclassification of the nonparametric procedure when the sample size is small and compare the results with the linear discriminant function probabilities of misclassification. The populations are assumed to be normal with equal covariance so the linear discriminant function is optimal and the nonparametric procedure can be compared against the optimum for a comparison as to how much discriminating power is lost when the sample sizes are small.

This Section will present some of the concepts and results of [3] developed in establishing this comparison.

Initially, it is stated that the problem can be reduced by considering linear transformations on the observation space so that F and G will always have the identity covariance matrix, i.e., the p transformed measurements are independent in each population and that each measurement has unit variance. Also the expectation vector of F can be placed at the origin and that of G on the positive first axis. Thus only two parameters p and λ are required to specify the transformed populations where

λ = E (first coordinate of Y)

= distance between the means of the transformed populations.

For the linear discriminant function, P_1 and P_2 are unchanged by this transformation, hence there is no loss of generality. P_1 and P_2 for the nonparametric discriminators are also unaffected since such linear transformations map the totality of possible distance functions of the original space one to one into the totality of the new space.

The univariate normal case is considered in depth due to its computational simplicity both for the linear discriminant function and the nonparametric procedure.

Considering the linear discriminant function first, the univariate case eliminates matrix computation and allows the classification problem to be stated as; compute the arithmetic mean of the sample means, $\frac{\bar{X}+\bar{Y}}{2}$, and assign Z to the population whose sample mean lies on the side of $(\bar{X} + \bar{Y})/2$ as does z itself. An error is committed then if and only if;

$$Z > (\bar{X} + \bar{Y})/2 \quad \text{and } \bar{Y} > \bar{X}$$

or

$$Z < (\bar{X} + \bar{Y})/2 \quad \text{and } \bar{Y} < \bar{X}.$$

Hence,

$$P_1 = P(Z > (\bar{X}+\bar{Y})/2,\ \bar{Y} > \bar{X}) + P(Z < (\bar{X}+\bar{Y})/2,\ \bar{Y} < \bar{X})$$

and P_1 = P_2 due to the symetry of this particular problem. In [3] it is shown by defining new variables of X, Y and Z, the limiting value of P_1 as $n \to \infty$ is .5.

Considering now the nonparametric procedure, again n=m and the populations are univariate normal. The discriminator will be the case of k = 1, i.e., assign Z to the nearest sample (rule of the nearest neighbor). The distance function utilized to measure the degree of nearness will be;

$$\Delta\,(x,z) = \max_{i=1}^{p} |x_i - z_i|$$

This function, $\Delta\,(x,z)$, describes a hyper-cube in p space and is only one of many functions that could be used. In the case of p = 1, $\Delta\,(x,z)$ corresponds to Euclidean distance.

To arrive at the error probabilities a conditional probability $P_1(z)$ is introduced and defined as the probability that the nearest of the 2n sample observations to Z is a Y given that Z = z. Then

$$(1) \qquad P_1 = E_Z[P_1(z)] = \int_{-\infty}^{\infty} f_Z(z)\; P_1(z)\; dz$$

where the distribution of Z,

$$f_Z(z) = (1/\sqrt{2\pi})\, \exp\,(-\tfrac{1}{2}\; z^2),$$

is the distribution of the X's, F.

In order to calculate $P_1(z)$ in general the quantities $H_Z(\delta)$ and $K_Z(\delta)$ are defined as follows:

$$H_Z(\delta) = P\ (\text{the distance of X from } z < \delta\,)$$

$$H_Z(\delta) = P\ (\text{the distance of Y from } z < \delta\,).$$

The explicit forms of $H_Z(\delta)$ and $K_Z(\delta)$ and therefore P_1, will be dependent on the distance function utilized to express the nearness.

In [3] the distance function utilized is

$$\Delta(x,z) = \max_{i=1}^{p} |x_i - z_i|$$

and since the univariate case is considered, $H_Z(\delta)$ and $K_Z(\delta)$ can be expressed as:

$$H_Z(\delta) = P(|X-z| < \delta)$$

$$K_Z(\delta) = P(|Y-z| < \delta)$$

The formulation of $P_1(z)$ in terms of $H_Z(\delta)$ and $K_Z(\delta)$ can be done by considering the distance from z to each of the Y observations,

$$|Y_1-z|,\ |Y_2-z|,\ \cdots,\ |Y_n-z| .$$

Hence with the assumption of independence and of equiprobable events,

$$P(\max |Y_i-z| < \delta) = P(|Y_1-z| < \delta) \cdots$$

$$P(Y_n-z| < \delta) = (K_Z(\delta))^n.$$

Therefore,

$$P(\min |Y_i-z| > \delta) = P(|Y_1-z| > \delta) \cdots$$

$$P(|Y_n-z| > \delta) = (1-K_Z(\delta))^n$$

or

$$P(\min |Y_i-z| < \delta) = 1-(1-K_Z(\delta))^n.$$

The density of the minimum distance between Y and z is then

$$n\,(1-K_Z(\delta))^{n-1}\,dK_Z(\delta)\,.$$

Similarly the minimum distance between X_i and z can be treated with the result that

$$P(\min |X_i - z| > \delta) = (1-H_Z(\delta))^n.$$

Now, $P_1(z)$ = P (nearest observation to Z is a Y given that

$$Z=z) = \int_0^\infty P(\min |X_i - z| > \delta \mid \min |Y_i - z| = \delta)\, f_{\min|Y_i - z|}(\delta)\, d\delta.$$

In terms of $H_Z(\delta)$ and $K_Z(\delta)$ then,

$$(2)\quad P_1(z) = n\int_0^\infty (1-H_Z(\delta))^n\,(1-K_Z(\delta))^{n-1}\,dK_Z(\delta)\,.$$

Equations 1 and 2 form the basis for computations for the rule of nearest neighbor for any p and any distance function, the explicit form of $H_Z(\delta)$ and $K_Z(\delta)$ being dependent on the distance function.

Figures 1 and 2 illustrate the comparison of the linear discriminant function and the nonparametric discriminator error probabilities when the distance function is Δ, k = 1 and the populations are univariate normal.

It is shown in [3] that for large n and in general,

$$P_1 \cong E\left[\frac{g(z)}{f(z)+g(z)}\right] = \int_{-\infty}^{\infty} \frac{g(z)\,f(z)\,dz}{f(z)+g(z)}\,,$$

the value which by Schwartz's inequality cannot exceed ½ .

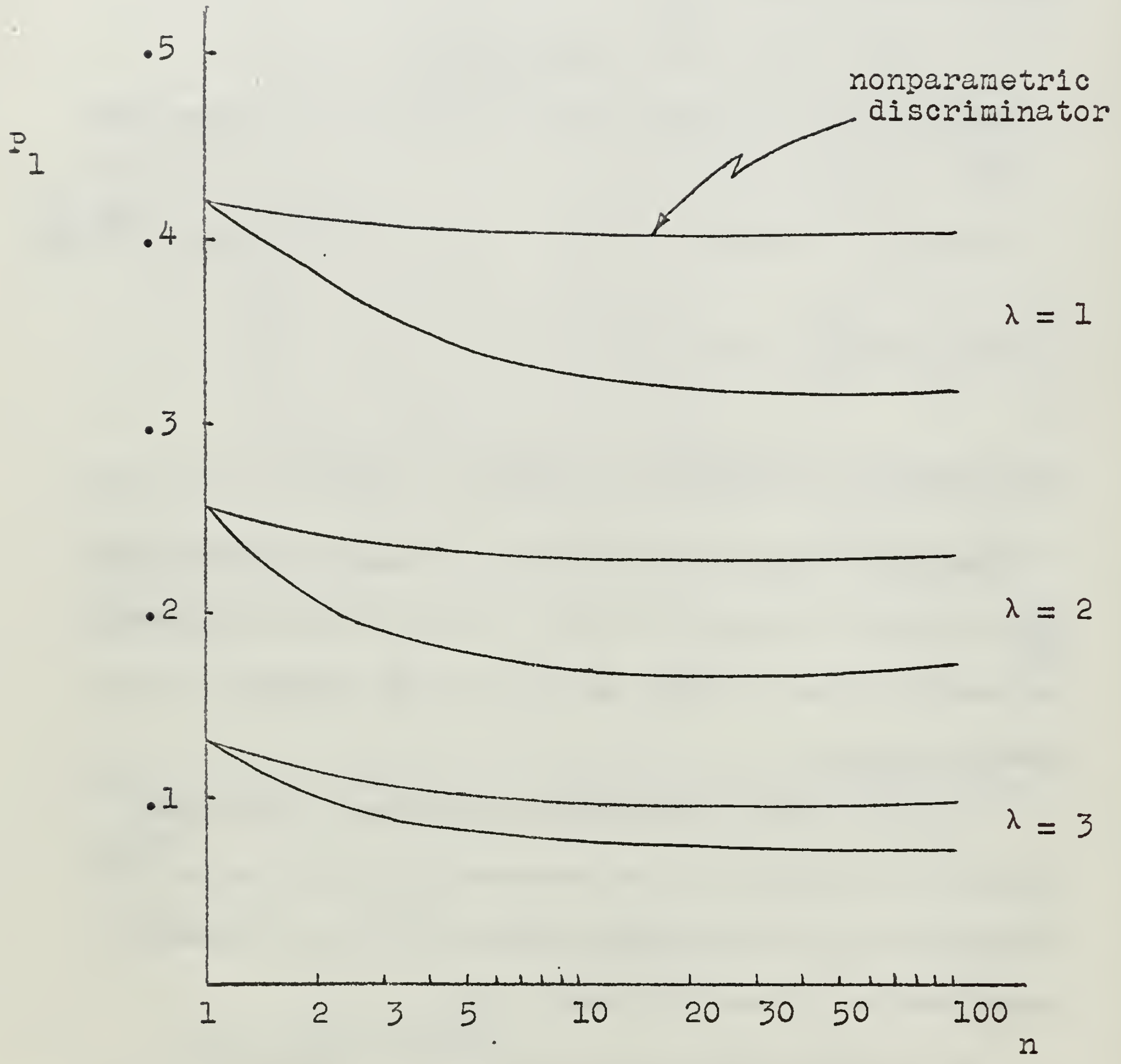

FIGURE 1
Probability of Error P_1 vs. n for Linear Discriminant Function and Nonparametric Discriminator, $k = 1$. Both populations are univariate normal. λ = distance between the means. Distance function is Δ .

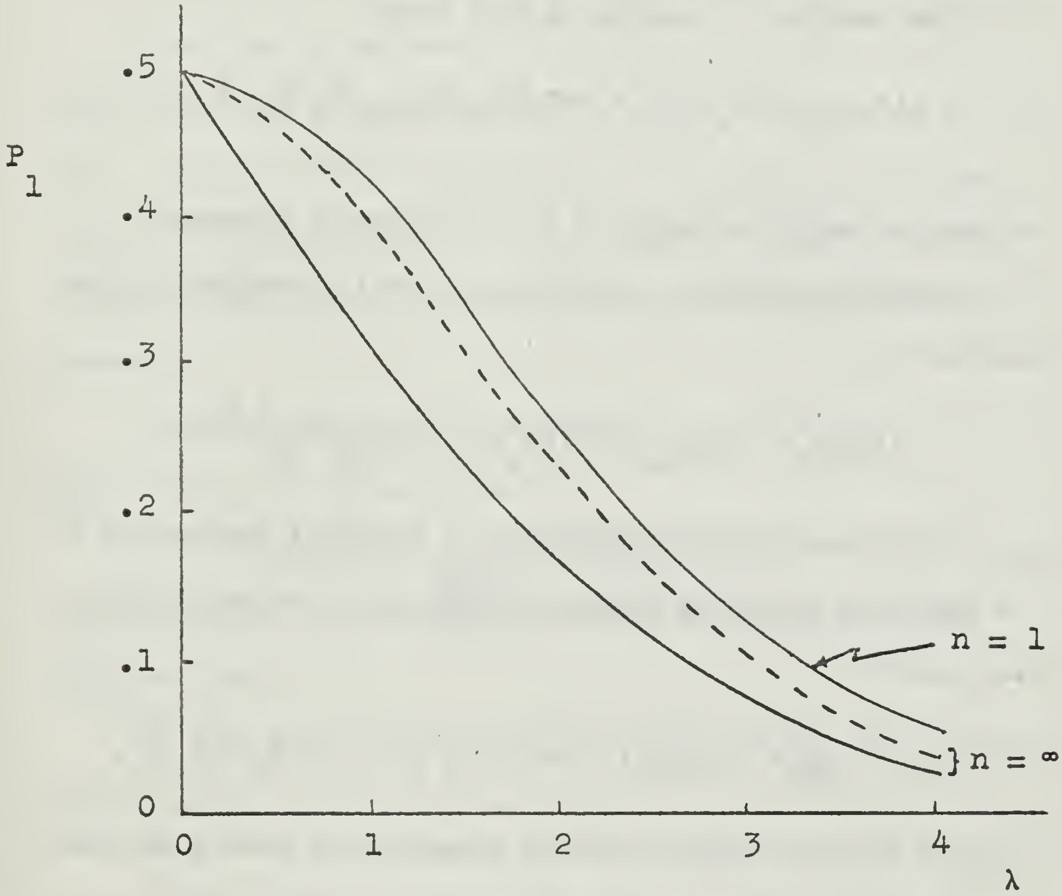

FIGURE 2

Probability of Error P_1 vs. λ for Linear Discriminant Function and Nonparametric Discriminator, $k = 1$. Both populations are univariate normal. Dotted line indicates nonparametric procedure. $n = 1$ is identical for both. Distance function is Δ .

The dependence of P_1 on the distance function was mentioned earlier and Hodges and Fix in [3] present some results showing how P_1 is affected by alternative distance functions.

In considering these other distance functions, the sample populations are assumed to be bivariate normal allowing a greater choice of distance functions than can be encountered when p=1.

The function $\Delta\ (\underline{x},\underline{z})$ is in this case,

$$\Delta\ ((x_1,x_2),\ (z_1,z_2)) = \max\ (|X_1-Z_1|\ ,\ |X_2-Z_2|\ )\ ,$$

a locus of points centered at z in the form of a square.

Euclidean distance, describing a circle centered at z is defined as,

$$\Delta_1\ ((x_1,x_2),\ (z_1,z_2)) = ((X_1-Z_1)^2 + (X_2-Z_2)^2)^{\frac{1}{2}}\ .$$

A distance function describing a rectangle centered at z in the ratio of one to three and whose sides are parallel to the axes is,

$$\Delta_2\ ((x_1,x_2),\ (z_1,z_2)) = \max\ (|X_1-Z_1|\ ,\ 3|X_2-Z_2|\ )\ ,$$

i.e., a rectangle whose vertical dimension is three times its horizontal dimension and the common multiple being the

$$\max\ (|X_1-Z_1|\ ,\ |X_2-Z_2|\ ).$$

Similarly a distance function describing a rectangle centered at z in the ratio of three to one and having sides parallel to the axes is,

$$\Delta_3\ ((x_1,x_2),\ (z_1,z_2)) = \max\ (3|X_1-Z_1|\ ,\ |X_2-Z_2|\ )\ .$$

Limited computed results are offered in [3] regarding the use of these distance functions Δ, Δ_1, Δ_2 and Δ_3. The comparative results are illustrated in Figure 3 in the form of a plot $P_1(0,0)$ and n. $P_1(0,0)$ is the conditional probability of error given that z is at the origin. It was presented because it was remarkably consistent with the value of P_1.

Comparison of the results in Figure 3 concludes that in this case there is little difference in P_1 whether Δ or Δ_1 is used. However there is great effect with respect to the use of the other distance functions and hence a burden is placed upon the statistician for selecting the appropriate distance function.

Though not covered in this summary, Hodges and Fix in [3] investigate to a limited extent:

(1) The nonparametric discriminator using Δ as a distance function with $k = 3$ for the univariate and bivariate normal distributions.

(2) The nonparametric discriminator using Δ as a distance function with $k = 1$, $n = 1$, and $p \geq 2$.

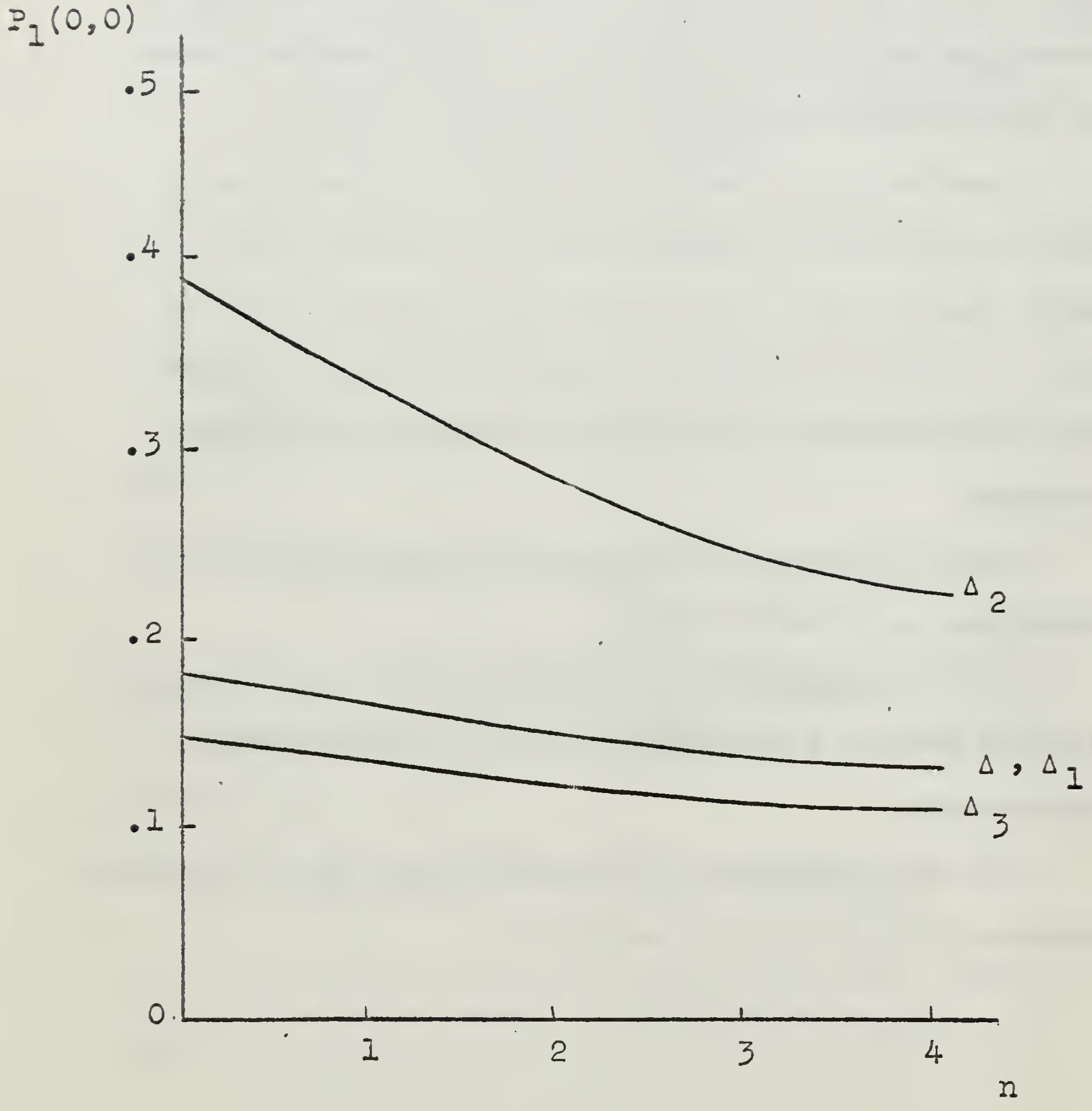

FIGURE 3

Probability of Error $P_1(0,0)$ vs. n for Various Distance Functions for the Nonparametric Procedure, k = 1. Both populations are bivariate normal, λ = 2.

3. NONPARAMETRIC CLASSIFICATION ERROR PROBABILITIES FOR TWO DIMENSIONAL EXPONENTIAL POPULATIONS FOR TWO DISTANCE FUNCTIONS

Hodges and Fix's work in [2] and [3] of comparing the performance of the linear discriminant function and that of the nonparametric procedure is applied to the two population exponentially distributed case by Eaton in [4] and Hager in [5]. Eaton studies the small sample performance and Hager extends Eaton's work to larger sample size and also gets some asymptotic results. In both [4] and [5] the study is limited to one dimension, i.e., $p = 1$. Hodges and Fix's major effort was for the case $p = 1$, with only limited presentation of results for $p \geq 2$ for the nonparametric procedure and none for the linear discriminant function.

This section will consider the following:

(1) The two population classification problem when the distributions are two dimensional exponential.

(2) Formulation of the error probability $P_1 = P(Z$ is assigned to $G | Z$ came from $F)$, utilizing the nonparametric procedure when the distance function is:

(a) Hyper-cube,

$$\Delta\,(\underline{x},\underline{z}) = \max_{i=1}^{p} [\,|x_i - z_i|\,]$$

(b) Hyper-sphere,

$$\Delta_1(\underline{x},\underline{z},) = [(x_1 - z_1)^2 + \cdots + (x_p - z_p)^2\,]^{\frac{1}{2}}$$

The density functions of F and G will be denoted by

$$f_{X_1X_2}(x_1,x_2) \text{ and } f_{Y_1Y_2}(y_1,y_2)$$

respectively where

$$f_{X_1X_2}(x_1,x_2) = \lambda_1\lambda_2\, e^{-(\lambda_1x_1 + \lambda_2x_2)} \qquad \text{for } x_i,\lambda_i \geq 0$$

$$= 0 \qquad \text{otherwise}$$

and

$$f_{Y_1Y_2}(y_1,y_2) = \mu_1\mu_2\, e^{-(\mu_1y_1 + \mu_2y_2)} \qquad \text{for } y_i,\mu_i \geq 0$$

$$= 0 \qquad \text{otherwise.}$$

Independence is assumed between X_1 and X_2 and also for Y_1 and Y_2.

Following the procedures of Section 2, $P_1 = E_Z[P_1(z)]$.

$$P_1 = \int_o^\infty \int_o^\infty P_1(z)\ f_{Z_1Z_2}(z_1,z_2)\ dz_1\ dz_2$$

$$P_1 = \int_{z_1 \geq z_2}\int P_1(z)\ f_{Z_1Z_2}(z_1,z_2)\ dz_2\ dz_1 + \int_{z_2 \geq z_1}\int P_1(z)\ f_{Z_1Z_2}(z_1,z_2) dz_2 dz_1$$

where

$$f_{Z_1Z_2}(z_1,z_2) = \lambda_1\lambda_2\, e^{-(\lambda_1z_1 + \lambda_2z_2)} \qquad \text{for } z_i,\lambda_i \geq 0$$

$$= 0 \qquad \text{otherwise} \quad .$$

$$P_1(z) = n\int_o^\infty [\,1 - H_Z(\delta)]^n\ [\,1 - K_Z(\delta)]^{n-1}\ dK_Z(\delta)$$

$$P_1(z) = n\int_{\delta < z_1 < z_2} [1-H_Z(\delta)]^n [1-K_Z(\delta)]^{n-1}\, dK_Z(\delta)$$

$$+ n\int_{z_1<\delta <z_2} [1-H_Z(\delta)]^n [1-K_Z(\delta)]^{n-1}\, dK_Z(\delta)$$

$$+ \int_{z_1<z_2< \delta} [1-H_Z(\delta)]^n [1-K_Z(\delta)]^{n-1}\, dK_Z(\delta)$$

when $z_2 \geq z_1$. When $z_1 \geq z_2$ the subscripts interchange.

The explicit form of $H_Z(\delta)$ and $K_Z(\delta)$ will be dependent on the distance function utilized and the relationship of δ and $\underline{z}$.

The hyper-cube distance function will be considered first,

$$\Delta\,(x,z) = \max_{i=1}^{2}\, [|X_i-z_i|]\ , \qquad \Delta\,(y.z) = \max_{i=1}^{2} [|Y_i-z_i|]\ .$$

For Z=z, and $\delta \geq 0$, $H_Z(\delta)$ and $K_Z(\delta)$ can be defined.

$$H_Z(\delta) = P(\max_{i=1}^{2} [|X_i-z_i|] < \delta\ ) = P(\max[|X_1-z_1|,\ |X_2-z_2|] < \delta\ )$$

$$= P(|X_1-z_1|,\ |X_2-z_2| < \delta\ ) = P(|X_1-z_1| < \delta\ )P(|X_2-z_2| < \delta),$$

assuming independence of the differences. Hence $H_Z(\delta)$ will be evaluated in six regions:

$$
\begin{aligned}
H_Z(\delta) &\left.\begin{array}{ll}
= \displaystyle\int_{z_1-\delta}^{z_1+\delta} f_{X_1}(x_1)\,dx_1 \int_{z_2-\delta}^{z_2+\delta} f_{X_2}(x_2)\,dx_2 & \delta \le z_2 \le z_1 \\[2ex]
= \displaystyle\int_{z_1-\delta}^{z_1+\delta} f_{X_1}(x_1)\,dx_1 \int_{0}^{z_2+\delta} f_{X_2}(x_2)\,dx_2 & z_2 \le \delta \le z_1 \\[2ex]
= \displaystyle\int_{0}^{z_1+\delta} f_{X_1}(x_1)\,dx_1 \int_{0}^{z_2+\delta} f_{X_2}(x_2)\,dx_2 & z_2 \le z_1 \le \delta
\end{array}\right\} z_1 \ge z_2 \\[3ex]
&\left.\begin{array}{ll}
= \displaystyle\int_{z_1-\delta}^{z_1+\delta} f_{X_1}(x_1)\,dx_1 \int_{z_1-\delta}^{z_2+\delta} f_{X_2}(x_2)\,dx_2 & \delta \le z_1 \le z_2 \\[2ex]
= \displaystyle\int_{0}^{z_1+\delta} f_{X_1}(x_1)\,dx_1 \int_{z_2-\delta}^{z_2+\delta} f_{X_2}(x_2)\,dx_2 & z_1 \le \delta \le z_2 \\[2ex]
= \displaystyle\int_{0}^{z_1+\delta} f_{X_1}(x_1)\,dx_1 \int_{0}^{z_2+\delta} f_{X_2}(x_2)\,dx_2 & z_1 \le z_2 \le \delta
\end{array}\right\} z_2 \ge z_1
\end{aligned}
$$

Similarly

$$K_Z(\delta) = P(|Y_1 - z_1| < \delta)\, P(|Y_2 - z_2| < \delta)$$

and will be defined as was $H_Z(\delta)$ in the six regions. Differentiating $K_Z(\delta)$ yields $dK_Z(\delta)$ in the six regions.

Evaluating $H_Z(\delta)$, $K_Z(\delta)$ and $dK_Z(\delta)$ and combining terms yields, when $z_1 \geq z_2$,

$$P_1(z) = n \int_0^{z_2} [1-4\, e^{-\lambda_1 z_1 - \lambda_2 z_2} \sinh\ \lambda_2 \delta \sinh\ \lambda_1 \delta]^n$$

$$[1-4\, e^{-\mu_1 z_1 - \mu_2 z_2} \sinh\ \mu_2\ \delta \sinh\ \mu_1\ \delta]^{n-1}$$

$$(4\, e^{-\mu_1 z_1 - \mu_2 z_2}) [\mu_1 \sinh\ \mu_2\ \delta \cosh\ \mu_1\ \delta$$

$$+ \mu_2 \sinh\ \mu_1\ \delta \cosh\ \mu_2\ \delta\ ]\ d\delta$$

$$+ n \int_{z_2}^{z_1} [1-2\, e^{-\lambda_1 z_1} \sinh\ \lambda_1\ \delta + 2\, e^{-\lambda_1 z_1 - \lambda_2 z_2 - \lambda_2\ \delta} \sinh\ \lambda_1 \delta]^n$$

$$[1-2\, e^{-\mu_1 z_1} \sinh\ \mu_1 \delta + 2\, e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_2 \delta} \sinh\ \mu_1 \delta]^{n-1}$$

$$[2\mu_1 e^{-\mu_1 z_1} \cosh\ \mu_1 \delta - 2\mu_1 e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_2 \delta}$$

$$\cosh\ \mu_1 \delta + 2\mu_2 e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_2 \delta} \sinh\ \mu_1 \delta\ ]\ d\delta$$

$$+ n \int_{z_1}^{\infty} [e^{-\lambda_1 z_1 - \lambda_1 \delta} + e^{-\lambda_2 z_2 - \lambda_2 \delta} - e^{-\lambda_1 z_1 - \lambda_2 z_2 - \lambda_1 \delta - \lambda_2 \delta}]^n$$

$$[e^{-\mu_1 z_1 - \mu_1 \delta} + e^{-\mu_2 z_2 - \mu_2 \delta} - e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta - \mu_2 \delta}]^{n-1}$$

$$[\mu_1 e^{-\mu_1 z_1 - \mu_1 \delta} + \mu_2 e^{-\mu_2 z_2 - \mu_2 \delta} - (\mu_1 + \mu_2) e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta - \mu_2 \delta}]\, d\delta$$

and when $z_2 \geq z_1$

$$P_1(z) = n\int_0^{z_1}[1-4\, e^{-\lambda_1 z_1-\lambda_2 z_2}\sinh\ \lambda_2\delta\ \sinh\ \lambda_1\delta]^n$$

$$[1-4\, e^{-\mu_1 z_1-\mu_2 z_2}\sinh\ \mu_2\delta\ \sinh\ \mu_1\delta]^{n-1}$$

$$[4\, e^{-\mu_1 z_1-\mu_2 z_2}(\mu_1\ \sinh\ \mu_2\delta\cos\ \mu_1\delta + \mu_2\sinh\ \mu_1\delta$$

$$\cosh\ \mu_2\delta)]d\delta$$

$$+\ n\int_{z_1}^{z_2}[1-2\, e^{-\lambda_2 z_2}\sinh\lambda_2\delta + 2e^{-\lambda_1 z_1-\lambda_2 z_2-\lambda_1\delta}\sinh\lambda_2\delta]^n$$

$$[1-2\, e^{-\mu_2 z_2}\sinh\mu_2\delta + 2e^{-\mu_1 z_1-\mu_2 z_2\ -\mu_1\delta}\sinh\mu_2\delta]^{n-1}$$

$$[2\mu_2 e^{-\mu_2 z_2}\cosh\mu_2\delta - 2\mu_2 e^{-\mu_1 z_1-\mu_2 z_2-\mu_1\delta}$$

$$\cosh\mu_2\delta + 2\mu_1 e^{-\mu_1 z_1-\mu_2 z_2-\mu_1\delta}\sinh\mu_2\delta]\ d\delta$$

$$+\ n\int_{z_2}^{\infty}[e^{-\lambda_1 z_1-\lambda_1\delta} + e^{-\lambda_2 z_2-\lambda_2\delta} - e^{-\lambda_1 z_1-\lambda_2 z_2-\lambda_1\delta-\lambda_2\delta}]^n$$

$$[e^{-\mu_1 z_1-\mu_1\delta} + e^{-\mu_2 z_2-\mu_2\delta} - e^{-\mu_1 z_1-\mu_2 z_2-\mu_1\delta-\mu_2\delta}]^{n-1}$$

$$[\mu_1 e^{-\mu_1 z_1-\mu_1\delta} + \mu_2 e^{-\mu_2 z_2-\mu_2\delta} - (\mu_1+\mu_2)\ e^{-\mu_1 z_1-\mu_2 z_2-\mu_1\delta-\mu_2\delta}]d\delta.$$

Then $P_1 = \int_0^{\infty} \lambda_1 e^{-\lambda_1 z_1} \int_0^{z_1} P_1(z)\, \lambda_2 e^{-\lambda_2 z_2}\, dz_2\, dz_1 +$

$$\int_0^{\infty} \lambda_2 e^{-\lambda_2 z_2} \int_0^{z_2} P_1(z) \lambda_1 e^{-\lambda_1 z_1}\, dz_1\, dz_2 \ .$$

The Euclidean distance or hyper-sphere distance function for $p = 2$ is:

$$\Delta_1(x,z) = [(x_1-z_1)^2 + (x_2-z_2)^2]^{\frac{1}{2}} \ , \ \Delta_1(y,z) = [(x_1-z_1)^2 + (x_2-z_2)^2]^{\frac{1}{2}} \ .$$

Evaluation of P_1, $P_1(z)$, $H_Z(\delta)$, $K_Z(\delta)$ and $dK_Z(\delta)$ will follow the same logic as that followed for the hyper-cube. The explicit form will be more awkward however.

Consider $H_Z(\delta) = P([(X_1-z_1)^2 + (X_2-z_2)^2]^{\frac{1}{2}} < \delta)$. For ease of notation define:

$$S = X_1 - z_1 \text{ and } T = X_2 - z_2 \ .$$

Then,

$$H_Z(\delta) = P(S^2 + T^2 \le \delta^2) = \iint_{S^2+T^2 \le \delta^2} f_{ST}(s,t)\, ds\, dt$$

where

$$f_S(s) = f_{X_1}(s+z_1) = \lambda_1 e^{-\lambda_1(s+z_1)} \qquad \text{for } s + z_1 \ge 0$$

$$= 0 \qquad \text{otherwise}$$

$$f_T(t) = f_{X_2}(t+z_2) = \lambda_2 e^{-\lambda_2(t+z_2)} \qquad \text{for } t + z_2 \ge 0$$

$$= 0 \qquad \text{otherwise.}$$

The definition of S and T translates the axes to center at (z_1, z_2) and $H_Z(\delta)$ is the evaluation of the circle centered at (z_1, z_2) with $\delta \ge 0$. Here as before, evaluation of $H_Z(\delta)$ will be over a region which is the union of six mutually exclusive sub-regions. Therefore,

$$H_Z(\delta) = \left.\begin{aligned} &4\int_0^{\delta}\int_0^{\sqrt{\delta^2-s^2}} f_{ST}(s,t)\,ds\,dt && \delta \le z_2 \le z_1 \\ &= \int_{-\delta}^{\delta}\int_{-z_2}^{\sqrt{\delta^2-s^2}} f_{ST}(s,t)\,ds\,dt && z_2 \le \delta \le z_1 \\ &= \int_{-z_1}^{\delta}\int_{-z_2}^{\sqrt{\delta^2-s^2}} f_{ST}(s,t)\,ds\,dt && z_2 \le z_1 \le \delta \end{aligned}\right\} z_1 \ge z_2$$

$$\left.\begin{aligned} &= 4\int_0^{\delta}\int_0^{\sqrt{\delta^2-s^2}} f_{ST}(s,t)\,ds\,dt && \delta \le z_1 \le z_2 \\ &= \int_{-z_1}^{\delta}\int_{-\sqrt{\delta^2-s^2}}^{\sqrt{\delta^2-s^2}} f_{ST}(s,t)\,ds\,dt && z_1 \le \delta < z_2 \\ &= \int_{-z_1}^{\delta}\int_{-z_2}^{\sqrt{\delta^2-s^2}} f_{ST}(s,t)\,ds\,dt && z_1 \le z_2 \le \delta \end{aligned}\right\} z_2 \ge z_1$$

$K_Z(\delta)$ is similarly treated by defining,

$$U = Y_1 - z_1 \ , \quad V = Y_2 - z_2 \quad .$$

$$K_Z(\delta) = P(U^2 + V^2 \le \delta^2) = \int\int_{U^2+V^2 < \delta^2} f_{UV}(u,v)\, dv\, du$$

where

$$f_U(u) = f_{Y_1}(u+z_1) = \mu_1 e^{-\mu_1(u+z_1)} \qquad \text{for } u + z_1 \ge 0$$

$$= 0 \qquad \text{otherwise}$$

and

$$f_V(v) = f_{Y_2}(v+z_2) = \mu_2 e^{-\mu_2(v+z_2)} \qquad \text{for } v + z_2 \ge 0$$

$$= 0 \qquad \text{otherwise.}$$

The functional form of $K_Z(\delta)$ will be the same as $H_Z(\delta)$. $K_Z(\delta)$ and $dK_Z(\delta)$ will be defined over the same regions as $H_Z(\delta)$. Evaluating $H_Z(\delta)$, $K_Z(\delta)$ and $dK_Z(\delta)$ and combining terms yields, when $z_1 \ge z_2$,

$$P_1(z) = n \int_0^{z_2} [1 + 4e^{-\lambda_1 z_1 - \lambda_2 z_2} - 4e^{-\lambda_1 z_1 - \lambda_2 z_2 - \lambda_1 \delta} - 4e^{-\lambda_1 z_1 - \lambda_2 z_2}$$

$$\int_0^{\delta} \lambda_1 e^{-\lambda_1 s - \lambda_2 \sqrt{\delta^2 - s^2}}\, ds]^n \, [1 + 4e^{-\mu_1 z_1 - \mu_2 z_2} - 4e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta}$$

$$- 4e^{-\mu_1 z_1 - \mu_2 z_2} \int_0^{\delta} \mu_1 e^{-\mu_1 u - \mu_2 \sqrt{\delta^2 - u^2}}\, du]^{n-1}$$

$$[-8\mu_1 e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta} + 4\mu_1 e^{-\mu_1 z_1 - \mu_2 z_2}$$

$$P_1(z) =$$

$$\int_0^{\delta} \frac{\mu_2\delta\, e^{-\mu_1 u-\mu_2\sqrt{\delta^2-u^2}}}{\sqrt{\delta^2-u^2}}\, du\ ]\, d\delta$$

$$+n\int_{z_2}^{z_1} [1 - 2\, e^{-\lambda_1 z_1} \sinh \lambda_1\delta + e^{-\lambda_1 z_1-\lambda_2 z_2} \int_{-\delta}^{\delta} e^{-\lambda_1 s-\lambda_2\sqrt{\delta^2-s^2}}\, ds]^n$$

$$[1 - 2\, e^{-\mu_1 z_1} \sinh \mu_1\delta + e^{-\mu_1 z_1-\mu_2 z_2} \int_{-\delta}^{\delta} e^{-\mu_1 u-\mu_2\sqrt{\delta^2-u^2}}\, du]^{n-1}$$

$$[2\mu_1\, e^{-\mu_1 z_1} \cosh \mu_1\delta - 2\, e^{-\mu_1 z_1-\mu_2 z_2} \cosh \mu_1\delta + \mu_2\delta\, e^{-\mu_1 z_1-\mu_2 z_2}$$

$$\int_{-\delta}^{\delta} \frac{e^{-\mu_1 u-\mu_2\sqrt{\delta^2-u^2}}}{\sqrt{\delta^2-u^2}}\, du]\, d\delta$$

$$+n\int_{z_1}^{\infty} [\, e^{-\lambda_1 z_1-\lambda_1\delta} + e^{-\lambda_1 z_1-\lambda_2 z_2} \int_{-z_1}^{\delta} e^{-\lambda_1 s-\ \lambda_2\sqrt{\delta^2-s^2}}\, ds]^n$$

$$[e^{-\mu_1 z_1-\mu_1\delta} + e^{-\mu_1 z_1-\mu_2 z_2} \int_{-z_1}^{\delta} e^{-\mu_1 u-\mu_2\sqrt{\delta^2-u^2}}\, du]^{n-1}$$

$$[\mu_1 e^{-\mu_1 z_1-\mu_1\delta} - e^{-\mu_1 z_1-\mu_2 z_2-\mu_1\delta} + \mu_2\delta\, e^{-\mu_1 z_1-\mu_2 z_2}$$

$$\int_{-z_1}^{\delta} \frac{e^{-\mu_1 u-\mu_2\sqrt{\delta^2-u^2}}}{\sqrt{\delta^2-u^2}}\, du]\, d\delta$$

and when $z_2 \geq z_1$,

$$P_1(z) = n \int_0^{z_1} [1 + 4\, e^{-\lambda_1 z_1 - \lambda_2 z_2} - 4e^{-\lambda_1 z_1 - \lambda_2 z_2 - \lambda_1 \delta} - 4e^{-\lambda_1 z_1 - \lambda_2 z_2}$$

$$\int_0^{\delta} \lambda_1 e^{-\lambda_1 s - \lambda_2 \sqrt{\delta^2 - s^2}} ds]^n$$

$$[1 + 4\, e^{-\mu_1 z_1 - \mu_2 z_2} - 4e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta} - 4e^{-\mu_1 z_1 - \mu_2 z_2}$$

$$\int_0^{\delta} \mu_1 e^{-\mu_1 u - \mu_2 \sqrt{\delta^2 - u^2}} du]^{n-1}$$

$$[-8\mu_1 e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta} + 4\mu_1 e^{-\mu_1 z_1 - \mu_2 z_2}$$

$$\int_0^{\delta} \frac{\mu_2 \delta\, e^{-\mu_1 u - \mu_2 \sqrt{\delta^2 - u^2}}}{\sqrt{\delta^2 - u^2}} du]\, d\delta$$

$$+ n \int_{z_1}^{z_2} [1 - 2\, \lambda_1 e^{-\lambda_1 z_1 - \lambda_2 z_2} \int_{z_1}^{\delta} e^{-\lambda_1 s} \sinh \sqrt{\delta^2 - s^2}\; ds]^n$$

$$[1 - 2\mu_1\, e^{-\mu_1 z_1 - \mu_2 z_2} \int_{z_1}^{\delta} e^{-\mu_1 u} \sinh \sqrt{\delta^2 - u^2}\; du]^{n-1}$$

$$[2\mu_1 \delta\, e^{-\mu_1 z_1 - \mu_2 z_2} \int_{z_1}^{\delta} \frac{e^{-\mu_1 u} \cosh \sqrt{\delta^2 - u^2}}{\sqrt{\delta^2 - u^2}} du]\, d\delta$$

$P_1(z) =$

$$+ n \int_{z_2}^{\infty} [e^{-\lambda_1 z_1 - \lambda_1 \delta} + e^{-\lambda_1 z_1 - \lambda_2 z_2} \int_{-z_1}^{\delta} e^{-\lambda_1 s - \lambda_2 \sqrt{\delta^2 - s^2}} ds]^n$$

$$[e^{-\mu_1 z_1 - \mu_1 \delta} + e^{-\mu_1 z_1 - \mu_2 z_2} \int_{-z_1}^{\delta} e^{-\mu_1 u - \mu_2 \sqrt{\delta^2 - u^2}} du]^{n-1}$$

$$[\mu_1 e^{-\mu_1 z_1 - \mu_1 \delta} - e^{-\mu_1 z_1 - \mu_2 z_2 - \mu_1 \delta} + \mu_2 \delta \, e^{-\mu_1 z_1 - \mu_2 z_2}$$

$$\int_{-z_1}^{\delta} \frac{e^{-\mu_1 u - \mu_2 \sqrt{\delta^2 - u^2}}}{\sqrt{\delta^2 - u^2}} du] \, d\delta$$

Then,

$$P_1 = \int_{z_1 \geq z_2}\int P_1(z) \, f_{Z_1 Z_2}(z_1, z_2) \, dz_1 dz_2 + \int_{z_2 \geq z_1}\int P_1(z) \, f_{Z_1 Z_2}(z_1, z_2) dz_1 dz_2$$

Representative values of P_1 were obtained for the hyper-cube distance function and are listed in Table 1. Evaluation was by a computer program utilizing FORTRAN 63 and a sub-routine based on Legendre-Gaussian quadrature. Appendix I lists the program as used. The case $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ was selected for comparison reasons and also for reduced computer time. The program is general and allows selection of any parameters greater than or equal to zero.

Time limitation prevented attainment of comparative results for the Euclidean distance function.

n	$\lambda_1 = \lambda_2 = 20$ $\mu_1 = \mu_2 = 10$	$\lambda_1 = \lambda_2 = 30$ $\mu_1 = \mu_2 = 10$	$\lambda_1 = \lambda_2 = 50$ $\mu_1 = \mu_2 = 10$
1	.3435	.2387	.1318
4	.3820	.2842	.1733
8	.3939	.2994	.1862
20	.4020	.3071	.1969

n	$\lambda_1 = \lambda_2 = 10$ $\mu_1 = \mu_2 = 20$	$\lambda_1 = \lambda_2 = 10$ $\mu_1 = \mu_2 = 30$	$\lambda_1 = \lambda_2 = 10$ $\mu_1 = \mu_2 = 40$
1	.5488	.5285	.5032
4	.4709	.3989	.3417
8	.4463	.3612	.2973
20	.4259	.3327	.2682

TABLE 1

Probability of Error P_1, Nonparametric Discriminator, Bivariate Exponential Distribution. Distance function, hyper-cube, Δ. $k = 1$, rule of nearest neighbor.

4. CONCLUSIONS AND ACKNOWLEDGEMENTS

The application of Hodges and Fix's work to the two population bivariate exponential case in Section 3 resulted in representative results tabulated in Table 1. Though the results are small in number, they allow comparison with that presented by Hager in [5].

The nonparametric discriminator error probability values in [5] indicate similarity to that listed in Table 1. The case when $\lambda_1 = \lambda_2 > \mu_1 = \mu_2$ corresponds to the case $c > 1$ in [5]. In both [5] and Table 1, probability of error P_1 increases with increase in n. Likewise $\lambda_1 = \lambda_2 < \mu_1 = \mu_2$ is analgous to $c < 1$ and P_1 decreases with increasing n in both Table 1 and [5]. Table 1 does not indicate the sensitivity of P_1 to change in parameter magnitudes nor the effect of the change of a single parameter.

The intent of this thesis was to develop the explicit forms of the error probabilities for the two distance functions and then evaluate the probabilities to determine if there is any superiority of one distance function over another. This goal was only partially attained due to the programming complexity and extensive computer time required to evaluate the error probability.

The following are recommended as areas of further work:

1. Examine sensitivity of P_1 to change in parameter magnitudes and variation of a single parameter.

2. Attempt to streamline program or approximation method to shorten computer time.

3. Compute P_1 for the Euclidean distance function to compare results with the hyper-cube.

4. Develop functional relationship of P_1 and P_2 when $p = 2$.

5. Examine other distance functions.

I wish to thank Professor J. R. Borsting for his enlightening guidance and assistance in preparing this thesis. I also wish to extend my gratitude to Mrs. Patricia Johnson for her programming assistance in evaluating the equations of Section 3 and to the Computer Facility for their help in processing the programs.

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APPENDIX I

```
      PROGRAM THRECUBE
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
    1 READ 1000,N,B1,B2,D1,D2
      PRINT2,N,B1,B2,D1,D2
    2 FORMAT(2X,2HN=I5,2X,3HB1=E15.8,2X,3HB2=E15.8,3X,3HD1=E15.8,2X,3HD2
     1=E15.8///)
 1000 FORMAT (I5,4F10.0)
      XN=N
      IND=1
      CALL TRAP 1(A1)
      PRINT3,A1
    3 FORMAT(5X,3HA1=E15.8///)
      IND=2
      CALL TRAP 1 (A2)
      PRINT 4,A2
    4 FORMAT(5X,3HA2=E15.8///)
      P1 = XN*(A1+A2)
      PRINT 2000,N,B1,B2,D1,D2,P1
 2000 FORMAT(2X,2HN=I5,2X,3HB1=E15.8,2X,3HB2=E15.8,2X,3HD1=E15.8,2X,
     1 3HD2=E15.8,2X,3HP1=E15.8///)
      GO TO 1
      END
      SUBROUTINE TRAP1(AREA)
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
      NN=10
      XINC = .5
      XEND = 0.
      AREA = 0.
   10 XI = XEND
      XEND = XEND +XINC
      AREAX = GLQUAD (XI,XEND,NN)
      AREA = AREA+AREAX
      IF(AREAX-1.E-06) 20,20,10
   20 RETURN
      END
      FUNCTION F(U)
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
      NN=10
      XINC =.5
      AREA = 0.
      XEND = 0.
      IF(ABSF(U)-1.E-06) 30,30,10
   10 XI=XEND
      XEND = XEND +XINC
      IF(XEND-U)15,15,12
   12 XEND = U
   15 AREAX = GLQUAD2 (XI,XEND,NN,U)
      AREA =AREA + AREAX
      IF(ABSF(U-XEND)-1.E-06) 30,30,20
   20 IF(AREAX-1.E-06) 30,30,10
   30 GO TO (40,50),IND
   40 B=B1
      GO TO 60
   50 B=B2
   60 F = B*EXPF(-B*U)*AREA
      RETURN
```

```
      END
      FUNCTION G(V,U)
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
      IMD=1
      CALL TRAP2 (0,V,U,V,A1)
      IMD=2
      CALL TRAP2 (V,U,U,V,A2)
      IMD=3
      CALL TRAP 3( U,U,V,A3)
      GO TO (400,500),IND
  400 B=B1
      GO TO 600
  500 B=B2
  600 G = B* EXPF(-B*V)*(A1+A2+A3)
      RETURN
      END
      SUBROUTINE TRAP2(A,C,U,V,AREA)
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
      NN=10
      XINC=.5
      AREA=0.
      XEND=A
      IF(ABSF(C)-1.E-06)30,30,10
   10 XI=XEND
      XEND = XEND +XINC
      IF(XEND-C)15,15,12
   12 XEND = C
   15 AREAX = GLQUAD 3 (XI,XEND,NN,U,V)
      AREA=AREA+AREAX
      IF(ABSF(C-XEND)-1.E-06)30,30,20
   20 IF(AREAX-1.E-06)30,30,10
   30 RETURN
      END
      SUBROUTINE TRAP3(A,U,V,AREA)
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
      NN=10
      XINC=.5
      AREA=0.
      XEND=A
   10 XI=XEND
      XEND = XEND +XINC
      AREAX = GLQUAD 3 (XI,XEND,NN,U,V)
      AREA=AREA+AREAX
      IF(AREAX-1.E-06)20,20,10
   20 RETURN
      END
      FUNCTION HF(X,U,V)
      COMMON B1,B2,D1,D2,N,XN,IND,IMD
      GO TO ( 100,500),IND
  100 Z1=U
      Z2=V
      GO TO (200,300,400),IMD
  200 HF= (1.-4.*EXPF(-B1*Z1-B2*Z2)* SINH(X,B2)*SINH(X,B1))**N*(1.-4.*
     1EXPF( -D1*Z1-D2*Z2)* SINH(D2,X)*SINH(D1,X))**(N-1)*(4.* EXPF(-D1*
     2Z1-D2*Z2)*(D1*SINH(D2,X)*COSH(D1,X)+D2*SINH(D1,X)*COSH(D2,X)))
      GO TO 900
```

```
300 HF = (1. -2.*EXPF(-B1*Z1)* SINH(B1,X)+2.*EXPF(-B1*Z1-B2*Z2-B2*X)
   1* SINH(B1,X))**N*(1.- 2.*EXPF(-D1*Z1)*SINH(D1,X) +2.*EXPF(-D1*Z1
   2-D2*Z2-D2*X) * SINH(D1,X))** (N-1)*( 2.*D1*EXPF(-D1*Z1)* COSH
   3 (D1,X) - 2.*D1*EXPF(-D1*Z1-D2*Z2 -D2*X) * COSH(D1,X) +2.*D2*EXPF
   4 (-D1*Z1-D2*Z2-D2*X) *SINH(D1,X))
    GO TO 900
400 HF= (EXPF(-B1*Z1-B1*X) +EXPF(-B2*Z2-B2*X) -EXPF(-B1*Z1-B2*Z2-B1*
   1 X-B2*X))**N* (EXPF(-D1*Z1-D1*X) +EXPF(-D2*Z2-D2*X) -EXPF(-D1*Z1-
   2D2*Z2 -D1*X-D2*X))**(N-1) *(D1*EXPF(-D1*Z1-D1*X) +D2* EXPF(-D2*
   3 Z2-D2*X)-(D1+D2) *EXPF(-D1*Z1-D2*Z2-D1*X-D2*X))
    GO TO 900
500 Z1=V
    Z2=U
    GO TO (600,700,800),IMD
600 HF= (1.-4.*EXPF(-B1*Z1-B2*Z2)* SINH(X,B2)*SINH(X,B1))**N*(1.-4.*
   1EXPF( -D1*Z1-D2*Z2)* SINH(D2,X)*SINH(D1,X))**(N-1)*(4.* EXPF(-D1*
   2Z1-D2*Z2)*(D1*SINH(D2,X)*COSH(D1,X)+D2*SINH(D1,X)*COSH(D2,X)))
    GO TO 900
700 HF=(1. -2.*EXPF(-B2*Z2)*SINH(B2,X)+2.*EXPF(-B1*Z1-B2*Z2-B1*X)*
   1SINH(B2,X))**N*(1.-2.* EXPF(-D2*Z2)* SINH(D2,X)+ 2.* EXPF(-D1*Z1-D
   22*Z2-D1*X) *SINH(D2,X))**(N-1) *(2.*D2*EXPF(-D2*Z2)* COSH(D2,X)
   3 -2.* D2*EXPF(-D1*Z1-D2*Z2-D1*X)*COSH(D2,X) +2.*D1 *EXPF(-D1*Z1-D2
   4*Z2 -D1*X)* SINH(D2,X))
    GO TO 900
800 HF= (EXPF(-B1*Z1-B1*X) +EXPF(-B2*Z2-B2*X) -EXPF(-B1*Z1-B2*Z2-B1*
   1 X-B2*X))**N* (EXPF(-D1*Z1-D1*X) +EXPF(-D2*Z2-D2*X) -EXPF(-D1*Z1-
   2D2*Z2 -D1*X-D2*X))**(N-1) *(D1*EXPF(-D1*Z1-D1*X) +D2* EXPF(-D2*
   3 Z2-D2*X)-(D1+D2) *EXPF(-D1*Z1-D2*Z2-D1*X-D2*X))
900 RETURN
    END
    FUNCTION COSH(Y,X)
    COSH=(EXPF(Y*X)+EXPF(-Y*X))/2.
    RETURN
    END
    FUNCTION SINH(Y,X)
    SINH=(EXPF(Y*X)-EXPF(-Y*X))/2.
    RETURN
    END
    FUNCTION GLQUAD (A,B,N)
C   D1 UCSD GLQUAD                                                      F
C   GAUSSIAN-LEGENDRE QUADRATURE OF F FROM A TO B, 10,20 OR 40 NODES.
C
C   P. YAGER 10/20/64              (REF. KRYLOV PP338,341 AND SEC 7.2)
C
    COMMON /GLQDATA/X1(5),A1(5),X2(10),A2(10),X4(20),A4(20)
    DATA (X1=
   D .9739065285, .8650633667, .6794095683, .4333953941, .1488743390)
    DATA (A1=
   D .0666713443, .1494513491, .2190863625, .2692667193, .2955242247)
    DATA (X2=
   D .9931285992, .9639719273, .9122344283, .8391169718, .7463319065,
   D .6360536807, .5108670020, .3737060887, .2277858511, .0765265211)
    DATA (A2=
   D .0176140071, .0406014298, .0626720483, .0832767416, .1019301198,
   D .1181945320, .1316886384, .1420961093, .1491729865, .1527533871)
    DATA (X4=
```

```
 D .9982377097, .9907262387, .9772599500, .9579168192, .9328128083,
 D .9020988070, .8659595032, .8246122308, .7783056514, .7273182552,
 D .6719566846, .6125538897, .5494671251, .4830758017, .4137792044,
 D .3419940908, .2681521850, .1926975807, .1160840707, .0387724175)
  DATA (A4=
 D .0045212771, .0104982845, .0164210584, .0222458492, .0279370070,
 D .0334601953, .0387821680, .0438709082, .0486958076, .0532278470,
 D .0574397691, .0613062425, .0648040135, .0679120458, .0706116474,
 D .0728865824, .0747231691, .0761103619, .0770398182, .0775059480)
  T0=(A+B)/2. $ T1=(B-A)/2. $ Y=0
  IF(N-10)1,1,4
1 DO 2 K=1,5
2 Y=Y+A1(K)*(F(T0-T1*X1(K))+F(T0+T1*X1(K)))
3 GLQUAD=Y*T1
  RETURN
4 IF(N-20)5,5,7
5 DO 6 K=1,10
6 Y=Y+A2(K)*(F(T0-T1*X2(K))+F(T0+T1*X2(K)))
  GOTO 3
7 DO 8 K=1,20
8 Y=Y+A4(K)*(F(T0-T1*X4(K))+F(T0+T1*X4(K)))
  GOTO 3
  END
  FUNCTION GLQUAD2 (A,B,N,U)
C     D1 UCSD GLQUAD
C     GAUSSIAN-LEGENDRE QUADRATURE OF F FROM A TO B, 10,20 OR 40 NODES.
C
C     P. YAGER 10/20/64              (REF. KRYLOV PP338,341 AND SEC 7.2)
C
  COMMON /GLQDATA/X1(5),A1(5),X2(10),A2(10),X4(20),A4(20)
  DATA (X1=
 D .9739065285, .8650633667, .6794095683, .4333953941, .1488743390)
  DATA (A1=
 D .0666713443, .1494513491, .2190863625, .2692667193, .2955242247)
  DATA (X2=
 D .9931285992, .9639719273, .9122344283, .8391169718, .7463319065,
 D .6360536807, .5108670020, .3737060887, .2277858511, .0765265211)
  DATA (A2=
 D .0176140071, .0406014298, .0626720483, .0832767416, .1019301198,
 D .1181945320, .1316886384, .1420961093, .1491729865, .1527533871)
  DATA (X4=
 D .9982377097, .9907262387, .9772599500, .9579168192, .9328128083,
 D .9020988070, .8659595032, .8246122308, .7783056514, .7273182552,
 D .6719566846, .6125538897, .5494671251, .4830758017, .4137792044,
 D .3419940908, .2681521850, .1926975807, .1160840707, .0387724175)
  DATA (A4=
 D .0045212771, .0104982845, .0164210584, .0222458492, .0279370070,
 D .0334601953, .0387821680, .0438709082, .0486958076, .0532278470,
 D .0574397691, .0613062425, .0648040135, .0679120458, .0706116474,
 D .0728865824, .0747231691, .0761103619, .0770398182, .0775059480)
  T0=(A+B)/2. $ T1=(B-A)/2. $ Y=0
  IF(N-10)1,1,4
1 DO 2 K=1,5
2 Y=Y+A1(K)*(G(T0-T1*X1(K),U)+G(T0+T1*X1(K),U))
3 GLQUAD2 =Y*T1
  RETURN
```

```
    4 IF(N-20)5,5,7
    5 DO 6 K=1,10
    6 Y=Y+A2(K)*(G(T0-T1*X2(K),U)+G(T0+T1*X2(K),U))
      GOTO 3
    7 DO 8 K=1,20
    8 Y=Y+A4(K)*(G(T0-T1*X4(K),U)+G(T0+T1*X4(K),U))
      GOTO 3
      END
      FUNCTION GLQUAD3 (A,B,N,U,V)
C     D1 UCSD GLQUAD                                                     F
C     GAUSSIAN-LEGENDRE QUADRATURE OF F FROM A TO B, 10,20 OR 40 NODES.
C
C     P. YAGER 10/20/64            (REF. KRYLOV PP338,341 AND SEC 7.2)
C
      COMMON /GLQDATA/X1(5),A1(5),X2(10),A2(10),X4(20),A4(20)
      DATA (X1=
     D .9739065285, .8650633667, .6794095683, .4333953941, .1488743390)
      DATA (A1=
     D .0666713443, .1494513491, .2190863625, .2692667193, .2955242247)
      DATA (X2=
     D .9931285992, .9639719273, .9122344283, .8391169718, .7463319065,
     D .6360536807, .5108670020, .3737060887, .2277858511, .0765265211)
      DATA (A2=
     D .0176140071, .0406014298, .0626720483, .0832767416, .1019301198,
     D .1181945320, .1316886384, .1420961093, .1491729865, .1527533871)
      DATA (X4=
     D .9982377097, .9907262387, .9772599500, .9579168192, .9328128083,
     D .9020988070, .8659595032, .8246122308, .7783056514, .7273182552,
     D .6719566846, .6125538897, .5494671251, .4830758017, .4137792044,
     D .3419940908, .2681521850, .1926975807, .1160840707, .0387724175)
      DATA (A4=
     D .0045212771, .0104982845, .0164210584, .0222458492, .0279370070,
     D .0334601953, .0387821680, .0438709082, .0486958076, .0532278470,
     D .0574397691, .0613062425, .0648040135, .0679120458, .0706116474,
     D .0728865824, .0747231691, .0761103619, .0770398182, .0775059480)
      T0=(A+B)/2. $ T1=(B-A)/2. $ Y=0
      IF(N-10)1,1,4
    1 DO 2 K=1,5
    2 Y=Y+A1(K)*(HF(T0-T1*X1(K),U,V)+HF(T0+T1*X1(K),U,V))
    3 GLQUAD3 =Y*T1
      RETURN
    4 IF(N-20)5,5,7
    5 DO 6 K=1,10
    6 Y=Y+A2(K)*(HF(T0-T1*X2(K),U,V)+HF(T0+T1*X2(K),U,V))
      GOTO 3
    7 DO 8 K=1,20
    8 Y=Y+A4(K)*(HF(T0-T1*X4(K),U,V)+HF(T0+T1*X4(K),U,V))
      GOTO 3
      END
         END
         FINIS
-EXECUTE.
```

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An approach to the classification problem is one that is dependent of the amount of information assumed to be known about the distributions of the populations. It is assumed in this thesis that nothing is known about the distributions for a two population case. The probability of misclassification of an individual Z is presented in general. The approach is carried further to develop explicit forms of the error probability when the two populations are bivariate exponential.

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