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ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

W. E. BLEICK AND PETER C. C. WANG¹

ABSTRACT. A complete asymptotic development of the Stirling numbers $S(N, K)$ of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans. Roy. Soc. Canad., 49 (1955), 49-54] and de Bruijn [*Asymptotic methods in analysis*, North-Holland, Amsterdam, 1958, pp. 102-109] for the asymptotic representation of the related Bell numbers.

1. **Introduction.** Hsu [1] has given the asymptotic expansion

$$(1) \quad S(N, K) \sim (\frac{1}{2}K^2)^{N-K} \left[1 + \sum_{s=1}^t K^{-s} f_s(N-K) + O(K^{-t-1}) \right] / (N-K)!$$

for Stirling numbers $S(N, K)$ of the second kind, where $f_s(N-K)$ are polynomials of argument $N-K$ and $f_s(0)=0$. The expansion (1) is useful only for $N-K$ small, as is indicated in §3. We obtain a complete asymptotic expansion of $S(N, K)$ in powers of $(N+1)^{-1}$, using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers. Convergence is demonstrated for $K < (N+1)^{2/3} / [\pi + (N+1)^{-1/3}]$. The expansion when divergent is still useful when used as an asymptotic series.

2. **Asymptotics of $S(N, K)$.** A generating function for $S(N, K)$ is

$$(2) \quad \left(\frac{e^z - 1}{z} \right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N, K) z^{N-K}.$$

Hence the Cauchy integral formula gives

$$(3) \quad S(N, K) = \frac{N!}{2\pi i K!} \int_C (e^z - 1)^K z^{-N-1} dz$$

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where the contour C encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$(4) \quad (t - z)e^{z-t} = te^{-t},$$

where $t=(N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z=u$ is on the positive real axis with $t-1 < u < t$. The quadratic approximation to xe^{-x} at $x=1$ shows that $u \approx 2/N$ for $K=N$ and large N . There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z=u$. Since there are no other roots of (4) for $|t-z| \leq t-u$, we may apply the Lagrange inversion formula to obtain

$$(5) \quad u = t - \sum_{m=1}^{\infty} m^{m-1}(te^{-t})^m/m!$$

convergent for $t > 1$. Another form of (4) is the identity

$$(6) \quad K = (N + 1)(1 - e^{-u})/u$$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour C descending from $z=u$ is taken as the line $z=u+iy$, $|y| < \infty$, parallel to the imaginary axis. The modulus of the integrand in (3) is maximal at $z=u$ on this path, since both $(e^z-1)^K$ and z^{-N-1} have this property. The closed contour C is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N > 0$. The integral in (3) then becomes

$$(7) \quad i(e^u - 1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u + iy) dy$$

where

$$(8) \quad \psi(z) = K \ln[(e^z - 1)/(e^u - 1)] - (N + 1)\ln(z/u).$$

The contribution of the various parts of the $z=u+iy$ path to the integral must now be studied. As $|\exp \psi(z)| = \exp \operatorname{Re} \psi(z)$ we have to study

$$\operatorname{Re} \psi(u + iy) = K \ln[(e^{2u} - 2e^u \cos y + 1)^{1/2}/(e^u - 1)] - (N + 1)\ln(1 + y^2 u^{-2})^{1/2}.$$

We shall show that we can restrict ourselves essentially to the interval $|y| < \pi$. Since $1 + y^2 u^{-2} \geq 1 + \pi(2y - \pi)u^{-2}$ for $y \geq \pi$ we have

$$(e^u - 1)^K u^{-N-1} \left| \int_{\pi}^{\infty} \exp \psi(u + iy) dy \right| < \frac{u^{1-N}(e^u + 1)^K}{\pi(N - 1)(1 + \pi^2 u^{-2})^{N/2-1/2}}$$

which is of $O(N^{-N}e^N)$ for K small and of $O(2^N/\pi^N N)$ for K large. Since $\operatorname{Re} \psi(u+iy)$ is even, the part of the integral (7) for $|y| > \pi$ tends toward zero as $N \rightarrow \infty$. We now direct our attention to the interval $|y| < \pi$ where the saddle point at $y=0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y| < u$, is

$$(9) \quad \psi = -\frac{N+1}{2u} \left(\frac{1}{u} - \frac{1}{e^u - 1} \right) y^2 + (N+1) \sum_{j=1}^{\infty} \frac{(iy)^{j+2}}{(j+2)!} \left(\frac{d}{dz} \right)^{j+1} \left[\frac{1 - e^{-u}}{u(e^z - 1)} - \frac{1}{z} \right]_{z=u}$$

where the identity (6) has been used. We now make the substitutions

$$(10) \quad w = [(N+1)/2]^{1/2} [1 - u/(e^u - 1)]^{1/2} y/u$$

and

$$(11) \quad a_j = \frac{(iwu)^{j+2} (d/dz)^{j+1} [(1 - e^{-u})/u(l^z - 1) - 1/z]_{z=u}}{(j+2)! [\frac{1}{2} - \frac{1}{2}u/(e^u - 1)]^{j/2+1}}$$

to obtain

$$(12) \quad S(N, K) = B \int_{-\infty}^{\infty} \exp\{-w^2 + f[(N+1)^{-1/2}]\} dw$$

where

$$(13) \quad B = N!(e^u - 1)^K / \pi(2(N+1))^{1/2} K! u^N (1 + u/(1 - \exp u))^{1/2}$$

and f is the analytic continuation of

$$(14) \quad f[(N+1)^{-1/2}] = \sum_{j=1}^{\infty} a_j (N+1)^{-j/2}.$$

To find an upper bound to $|a_j|$ we note that $(e^z - 1)^{-1} = \sum_{k=1}^{\infty} e^{-kz}$ for $\operatorname{Re} z > 0$. Then

$$(d/du)^n (e^u - 1)^{-1} = (-1)^n \sum_{x=0}^{\infty} g(x)$$

where $g(x) = x^n e^{-ux}$. On using the Euler-Maclaurin sum formula [5] we find

$$\sum_{x=0}^{\infty} g(x) = \frac{n!}{u^{n+1}} + R_n.$$

The remainder $R_n = \int_0^{\infty} (x - [x] - \frac{1}{2}) g'(x) dx$ may be evaluated by a Laplace transform [6] to be

$$R_n = (-1)^{n-1} [nF^{(n-1)}(u) + uF^{(n)}(u)]$$

where $F(u) = u^{-2} - \frac{1}{2}u^{-1} \coth \frac{1}{2}u$. We conclude that $|R_n| \ll n!/u^{n+1}$ for small u and tends to zero for large u . Since $(1 - e^{-u})/u$ is less than unity we have

$$(15) \quad |a_j| < \sigma^{j+2}/j$$

where

$$(16) \quad \sigma = w^{2^{1/2}}/(1 - u/(e^u - 1))^{1/2} = y(N + 1)^{1/2}/u.$$

Remembering that we need not integrate (7) beyond $|y| = \pi$ for large N , we see by (15) and (16) that the series (14) is convergent for $\pi/u < 1$. We now expand $\exp f[(N+1)^{-1/2}]$ in a Taylor series of the form

$$(17) \quad \exp f[(N + 1)^{-1/2}] = \sum_{j=0}^{\infty} b_j(N + 1)^{-j/2}$$

where $b_0 = 1$ and b_j are polynomials in w of the degree and parity of $3j$. By a lemma of Moser and Wyman [4]

$$(18) \quad |b_j| \leq \sigma^{j+2}(1 + \sigma^2)^{j-1}.$$

Using (17) we may write (12) in the form

$$S(N, K) = B \left[\sum_{j=0}^{s-1} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} dw + R_s \right].$$

The absolute value of the remainder R_s is found from (18) to be

$$(19) \quad |R_s| \leq (N + 1)^{-s} \int_{-\infty}^{\infty} e^{-w^2} P_s(|w|) dw/M$$

where $P_s(|w|)$ is a polynomial in $|w|$ and

$$M = 1 - \sigma^2(1 + \sigma^2)^2/(N + 1).$$

On limiting the integration in (7) to $|y| < \pi$ we see that the remainder R_s exists if

$$(\pi/u)[1 + (N + 1)(\pi/u)^2] < 1.$$

Since $u + 1 > (N + 1)/K$ convergence occurs for

$$K < (N + 1)^{2/3}/[\pi + (N + 1)^{-1/3}]$$

approximately. For these values of K we conclude that

$$(20) \quad S(N, K) \sim B \left\{ \sum_{j=0}^{s-1} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} dw + O[(N + 1)^{-s}] \right\}.$$

The first two terms of (20) have been calculated to be

$$(21) \quad S(N, K) \sim \frac{N!(e^u - 1)^K}{(2\pi(N + 1))^{1/2} K! u^N (1 - G)^{1/2}} \cdot \left[1 - \frac{2 + 18G - 20G^2(e^u + 1)}{24(N + 1)(1 - G)^3} + \frac{3G^3(e^{2u} + 4e^u + 1) + 2G^4(e^{2u} - e^u + 1)}{24(N + 1)(1 - G)^3} \right]$$

where by [5]

$$G = u/(e^u - 1) = 1 - \frac{1}{2}u + \sum_{k=1}^{\infty} B_{2k} u^{2k}/(2k)!$$

The bracketed expression in (21), argumented by an additional inverse power of $N + 1$, is approximated by

$$1 - \frac{1}{6u(N + 1)} + \frac{1}{72u^2(N + 1)^2}$$

for small u and by

$$1 - \frac{1}{12(N + 1)} + \frac{1}{288(N + 1)^2}$$

for large u . These are the leading terms of an alternating asymptotic series.

3. Numerical example. The 6-significant-figure Table 1 compares the exact values of $S(100, K)$ with the values computed from (20) and (1) for several K . The excellent results obtained from (20) for values of K outside the interval of convergence show that the expansion gives useful results when used as an asymptotic series.

TABLE 1

K	$S(100, K)$ Exact	$S(100, K)$ 1 term of (20)	$S(100, K)$ 2 terms of (20)	$S(100, K)$ 4 terms of (1)
2	6.33825 10 ²⁹	6.34348 10 ²⁹	6.33825 10 ²⁹	1.81186 10 ⁻¹¹⁶
25	2.58320 10 ¹¹⁴	2.58496 10 ¹¹⁴	2.58321 10 ¹¹⁴	2.94696 10 ⁸³
50	4.30983 10 ¹⁰¹	4.30900 10 ¹⁰¹	4.30977 10 ¹⁰¹	1.51529 10 ⁹⁴
75	1.82584 10 ⁸³	1.82671 10 ⁸³	1.82579 10 ⁸³	5.32626 10 ⁸²
99	4.95000 10 ³	5.14199 10 ³	4.94451 10 ³	4.95000 10 ³

REFERENCES

1. L. C. Hsu, *Note on an asymptotic expansion of the nth difference of zero*, Ann. Math. Statist. **19** (1948), 273-277. MR **9**, 578.

2. Leo Moser and Max Wyman, *An asymptotic formula for the Bell numbers*, Trans. Roy. Soc. Canad. Sect. III (3) **49** (1955), 49–54. MR **17**, 1201.
3. N. G. de Bruijn, *Asymptotic methods in analysis*, Bibliotheca Mathematica, vol. 4, North-Holland, Amsterdam; Groningen and Interscience, New York, 1958, pp. 102–109. MR **20** #6003.
4. Leo Moser and Max Wyman, *On solutions of $X^d=1$ in symmetric groups*, Canad. J. Math. **7** (1955), 159–168. MR **16**, 904.
5. Konrad Knopp, *Theory and applications of infinite series*, Blackie and Son, London, 1928, pp. 523–528.
6. G. E. Roberts and H. Kaufman, *Table of Laplace transforms*, Saunders, Philadelphia, 1966, p. 137. MR **32** #8050.

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