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Operational availability with limited spares support

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NAVAL POSTGRADUATE SCHOOL
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THESIS

Operational Availability with
Limited Spares Support

by

Choi, Boo Woong

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Operational Availability with Limited Spares Support

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March 1977

ABSTRACT

Mathematical expressions are derived for the operational availability of a system having a finite number of standby spare components. The availability expressions are then used in a decision criterion for the selection of items to be stocked to support units on extended missions with no resupply capability. Various algorithms for determining how best to allocate constrained resources so as to maximize system operational availability are examined. Comparisons with the current Navy shipboard allowance list policy reveal that large potential savings in resources or better supply effectiveness could result from implementation of the procedures of this report.

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TABLE OF SYMBOLS AND ABBREVIATIONS

$A_k^{(i)}(t)$	-	Prob [ith equipment is "up" at time t given k spares available, and ith equipment is "up" at time zero] = Operational availability of ith redundant system without repair capability at time t, given k spares
$A_{sys}(t)$	-	Operational availability of weapon system at time t
C_v	-	Variable cost of a repair
$B_k(t)$	-	Prob [equipment is "up" at time t given k spares available, and equipment is "down" at time zero]
C_d		Cost of system being down per time
β	-	Constant repair rate
C_f	-	Fixed cost of repair
$C_{(i)}$	-	Unit spare cost of ith system for time t
$DC(t)$	-	Expected cost of system being down
$\Delta_k(t)$	-	Incremental benefit which can be obtained by adding an extra spare $[A_k(t)-A_{k-1}(t)]$
$\Delta_k^N(t)$	-	Value of $\Delta_k(t)$ obtainable by normal approximations
$\Delta_{sys}^{(i)}(t)$	-	Weapon system's incremental operational availability by adding a unit of spare of ith system to the weapon system
$F(t)$		CDR of time-to-failure random variable
$G(t)$		CDF of time-to-replacement random variable
$H(t)$		CDF of time-to-failure plus time-to-replacement random variable
k		Number of spare parts
MTTF		Mean time to failure
MTTR		Mean time to repair
MTTC		Mean time to replacement

μ	-	Constant replacement repair
$N(t)$	-	Number of failures for time t
$Q_k(t)$	-	Operational availability of redundant system with repair capability at time t, given k spares
R	-	Replacement time random variable
λ	-	Constant failure rate
T	-	Time-to-failure random variable
$TC(k)$	-	Total cost of system, given k spares
γ	-	$(\mu-\lambda)$
θ		$(\lambda\mu)$

I. INTRODUCTION

Most supply systems encountered in the real world are multiechelon in nature and all levels contribute to the achievement of the support goals of the system. The most important support level viewed with the objective of attaining a high level of equipment operational availability is the last echelon of support, such as the shipboard echelon of support in the Naval Supply Systems. This echelon of support is responsible for enabling operating units to be self-supporting for reasonable periods of time.

The operational availability of an equipment for a certain period of time is generally conceptualized as "the proportion of the time an equipment will spend in acceptable states." [8] In connection with this, operational availability is often used as a measure of effectiveness in the complex multiechelon supply systems. The definition of operational availability for both components and systems which is most often used is "the ratio of the mean time between failure (MTBF) to the sum of MTBF, the mean time to repair (MTTR) (perhaps with a further partition of MTTR into the mean logistics delay time, supply response time and repair time)." [10] The steady state operational availability upon which the above definition is based is not correct, however, if the number of spare parts is finite.

The operational availability of an equipment (or a system) depends on various factors, such as the reliability of its components, the repair and replacement delays, the availability of spare parts and supply response times. The Naval Supply System influences the latter two variables through its resource allocation decisions and policies. The system attempts to achieve high availabilities of spare parts at the last echelon of support (ship storeroom) through its Coordinated Shipboard Allowance List (COSAL). The present stockage policy focuses on those components which, historically, have experienced sufficient demand to justify spare part support. If a component qualifies to be included in the COSAL, its depth is determined as that amount sufficient to satisfy, with a specified probability (the same probability for all components), all demands which might occur in a given interval of time.

The present COSAL procedure has a great deal of intuitive and computational appeal. Since it is demand based and historical demand data are available, the COSAL is simple to determine provided sufficient resources are available. However, the current procedure has some shortcomings:

A. Satisfying demands with a given probability for a given interval of time does not guarantee a high level of operational availability.

B. No attempt is made to look at system configurations in determining what components to support.

C. When resource constraints are active, the use of military essentiality codes and other "fixes" must often be made to augment COSAL rules to restrict the allowance lists.

Because of these and perhaps other deficiencies of the current COSAL policy, improved COSAL determination schemes need to be investigated.

This paper develops two probabilistic models for computing the operational availabilities of systems with spare parts and it examines several resource allocation algorithms. It offers an alternative procedure for allocating spare stock at the shipboard level which could be directly related to the operational availabilities of the weapon systems being supported, and which would alleviate the short-comings of the present COSAL procedure described above. Renewal theory and Markovian analysis techniques are employed to obtain the results.

In Chapter II, the thesis develops a mathematical expression for the operational availability of a redundant system where there is no repair capability. Mathematical expressions for computing increment availabilities by adding an extra spare part are also determined in Chapter II. In Chapter II another mathematical model is developed for the determination of the operational availability of redundant systems where a repair capability exists. For modeling simplicity throughout the thesis, the mean time to failure,

the mean time to repair, and the mean time to replacement are all assumed to have exponential distributions. The first section of Chapter IV addresses a decision rule which considers the cost of spares and the cost of repairs to determine whether an extra spare should be carried or if a repair capability should be bought. Another decision rule which considers the "cost of system being down" as well as the cost of spares and cost of repair is examined in the second section of Chapter IV to determine the optimum number of spares which should be carried. In the last section of Chapter IV an algorithm to determine how best to allocate a fixed budget is presented. That algorithm ignores system configuration, treating each component as a system. The algorithm attempts to maximize operational availability per dollar expended on spares support. Chapter V shows how the resource allocation scheme can be affected by considering the system configurations. Each chapter includes numerical examples to illustrate the mathematical results. Summary and conclusions of the paper are presented in Chapter V, together with suggested extensions of this study.

II. OPERATIONAL AVAILABILITY FOR A SYSTEM WITH STANDBY SPARES AND NO REPAIR CAPABILITY (TYPE-1 SYSTEM)

Mathematical expressions for computing the operational availability of a system having a finite number of spare components are derived in this chapter. It is assumed that no repair capability exists for the system, so that, after all spares are used, the system must remain down if a failure occurs. Mathematical expressions are also derived for the incremental change in operational availability for additions to the number of spares carried. These expressions form the basis for the work presented in later chapters.

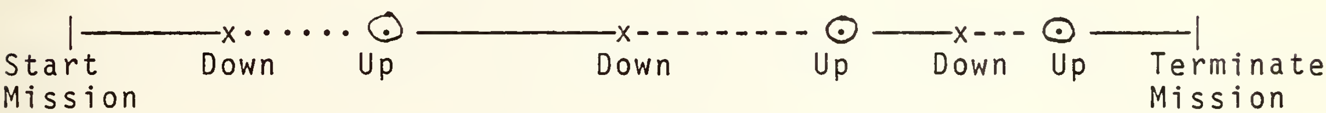
There have been many studies and papers written on standby redundancy in systems with repair maintenance. See, for example, [5, 9, 11, 12]. However, for the problem considered here, the systems often cannot be repaired at sea because of lack of a repair capability or other resources or because of constraints of space and time. Apparently, little work has been done for the case of finite spares and no repair capability.

A. MATHEMATICAL EXPRESSIONS FOR OPERATIONAL AVAILABILITY (TYPE-1 SYSTEM)

Consider a system of a single component having exponential times to failure and exponential times to replace with mean time between failures (MTBF) and mean time to change (MTTC) equal to $1/\lambda$ and $1/\mu$, respectively. Suppose that k identical spares are provisioned and available for support

at the start of a mission. When an operating unit fails, replacement begins immediately provided a spare unit is available. Assume that the time to failure of the i^{th} unit, T_i, and the time to change, R_i are independent, and the system under consideration is operational ("up") at time $t = 0$. Finally, suppose that non-operating idle spare parts are not subject to failure or deterioration. A system with these properties will be called a "Type-1 System."

For $k = \infty$, the up and down availability are repeated over the duration of the total mission of the ship. The availability status of the system can be described as an alternating renewal process shown in Figure 1.



OPERATIONAL STATUS OF SYSTEM WITH UNLIMITED SPARES

Figure 1

For this case, the operational availability of the system at time t is easily shown [1] to be:

$$A_\infty(t) = \frac{MTBF}{MTBF+MTTC} + \frac{MTTC}{MTBF+MTTC} \exp(-t(\frac{1}{MTBF} + \frac{1}{MTTC})) \qquad (1)$$

On taking the limit as $t\to\infty$, the steady state operational availability is found to be:

$$\lim_{t\to\infty} A_\infty(t) = \frac{MTBF}{MTBF+MTTC} \qquad (2)$$

Equation (2) is widely used as a definition of operational availability. However, as stated above, it is exact only for the case where the number of spares is unlimited. If the number of spare parts is finite, the operational status of the equipment is not described by an alternating renewal process. Once the spares are exhausted and the system goes down, it will remain down for the remaining duration of the ship's mission or perhaps the ship's mission must be aborted. Of course, if the number of spares is large or if the length of the mission is short, expression (2) should provide a good approximation to the true operational availability. In times of tight resources such as those currently experienced, it is not possible to maintain large pools of spare parts. Therefore, it is important to know just what is the availability for any finite number of spares. Resource allocation algorithms must know the benefits to be derived from incremental spare part allocations in order to allocate resources efficiently. Thus, expressions for the availability for any finite number of spares must be determined.

Define the following probabilities:

$A_k(t) = P_r$[system is up at t | k spares and system is up at time 0]

$B_k(t) = P_r$[system is up at t | k spares and system is down at time 0]

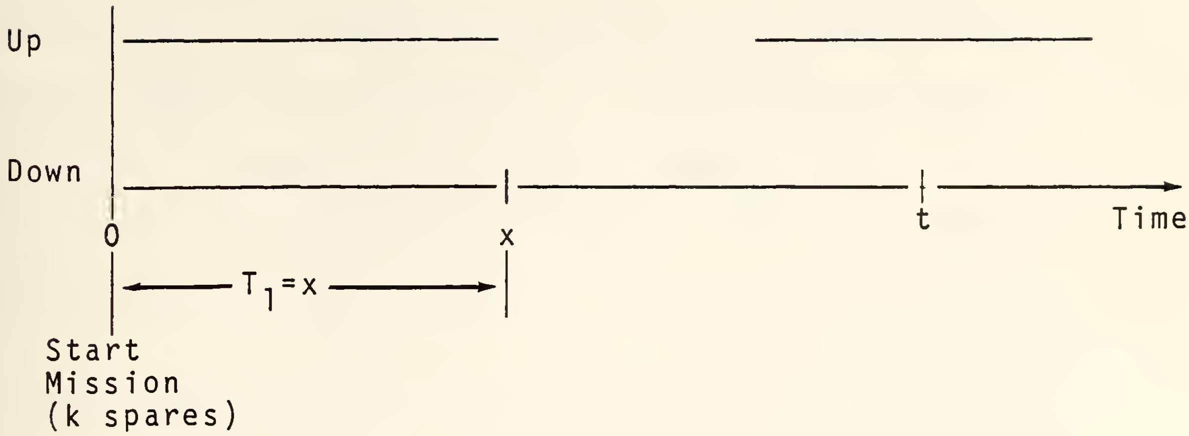


Figure 2. Operational Status with k Spares (Up at time 0)

Figure 2 illustrates a single cycle of the system with a finite number of spares. Now, it is important to consider as the "state" of the system not only its operational status, but also the number of spares available. The general formula for the operational availability at time t can be obtained by conditioning on the lifetime of the component, T_1. This yields the expression:

$$A_k(t) = \bar{F}(t) + \int_0^t B_k\ (t-x)dF(x) \tag{3}$$

where $F(x) = P_r[T \leq x]$. (The system is up at time t if the first unit survives to time t or if the system goes down at time $x<t$, and the system is back up and operating at time t.) Expression (1) can also be written as

$$A_k(t) = \bar{F}(t) + \int_0^t \bar{F}(t-x)dM_H(x)$$

where $H(x) = P_r[T+R \leq x]$ and $M_H(x)$ is the renewal function of $H(x)$. Now, let us see how $B_{k-1}(t)$ can be expressed in terms of $A_k(t)$. Suppose the system is down at time 0 and is restored to up condition at time x. This is shown in Figure 3.

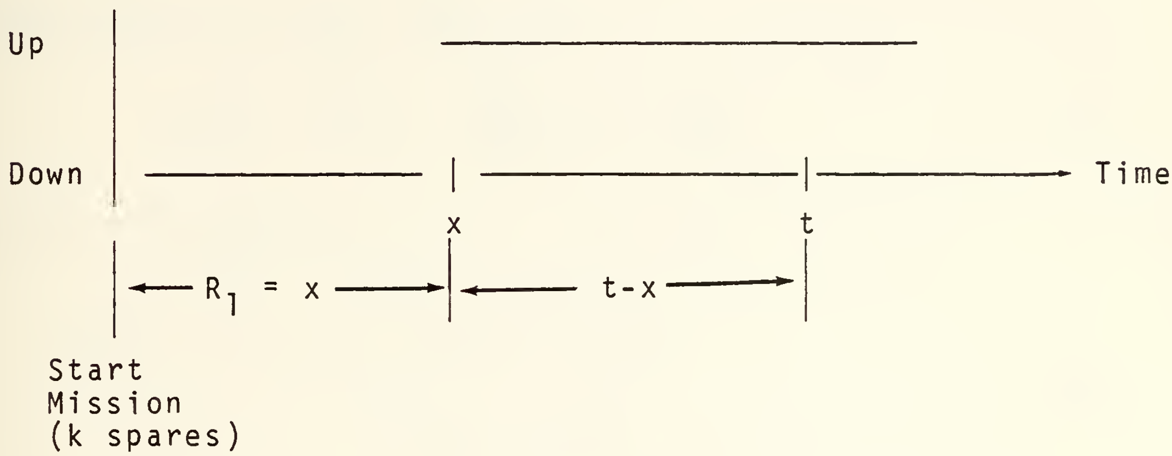


Figure 3. Operational Status with k Spares (System Down at Time 0)

Then, conditioned on $R_1 = x$,

$$B_k(t)/R_1 = x = \begin{cases} 0 & \text{for } x \geq t \\ A_{k-1}(t-x) & \text{for } x < t. \end{cases}$$

Therefore, $$B_k(t) = \int_0^t A_{k-1}(t-x)dG(x) \qquad (4)$$

where $G(x) = Pr[R \leq x]$. Substituting (4) into (3) gives

$$A_k(t) = F(t) + \int_0^t [\int_0^{t-x} A_{k-1}(t-x)dG(x)]dF(x) \qquad (5)$$

This can be simplified by taking the Laplace transform of both sides giving:

$$\widetilde{A_k(s)} = \widetilde{\bar{F}(s)} + \widetilde{B_k(s)} \quad \cdot \widetilde{F(s)} \tag{6}$$

and from (4) $\widetilde{B_k(s)} = \widetilde{A_{k-1}}(s) \cdot \widetilde{G(s)}$ (7)

Substituting (7) into (6) gives:

$$\widetilde{A_k(s)} = \widetilde{\bar{F}(s)} + \widetilde{A_{k-1}(s)} \cdot \widetilde{H(s)} \tag{8}$$

Inverting (8) gives

$$A_k(t) = \bar{F}(t) + \int_0^t A_{k-1}(t-x)dH(x) = \bar{F}(t) + \int_0^t A_{k-1}(t-x)h(x)dx \tag{9}$$

where h(x) is the density of H(x). Equation (9) is a general formula for computing the operational availability of a Type-1 System at an arbitrary time t for $k = 1,2,\cdots$. For k=0, it is clear that $A_o(t) = \bar{F}(t) = e^{-\lambda t}$. The expressions obtained from (9) for k = 0,1,3 and 4, and $t \geq 0$, are summarized below with $\theta = \lambda\mu$ and $\gamma = \mu - \lambda$:

$$A_o(t) = e^{-\lambda t} \tag{10}$$

$$A_1(t) = (1 - \frac{\theta}{\gamma^2} + \frac{\theta t}{\gamma})e^{-\lambda t} + \frac{\theta}{\gamma^2} e^{-\mu t} \tag{11}$$

$$A_2(t) = [1 - \frac{\theta}{\gamma^2} + \frac{3\theta^2}{\gamma^4} + (\frac{\theta}{\gamma} - \frac{2\theta^2}{\gamma^3})t + \frac{\theta^2}{2\gamma^2} t^2]e^{-\lambda t} + \frac{\theta}{\gamma^2} (1 - \frac{3\theta}{\gamma^2} - \frac{\theta}{\gamma})e^{-\mu t} \tag{12}$$

$$A_3(t) = [1 - \frac{\theta}{\gamma^2} + \frac{3\theta^2}{\gamma^4} - \frac{10\theta^3}{\gamma^6} + (\frac{\theta}{\gamma} - \frac{2\theta^2}{\gamma^3} + \frac{6\theta^3}{\gamma^5})\ t \tag{13}$$

$$+ (\frac{\theta^2}{2\gamma^2} - \frac{3\theta^3}{2\gamma^4})t^2 + \frac{\theta^3}{6\gamma^3}\ t^3]e^{-\gamma t} +$$

$$+ \frac{\theta}{\gamma^2}\ [1 - \frac{3\theta}{\gamma^2} + \frac{10\theta^2}{\gamma^4} - (\frac{\theta}{\gamma} - \frac{4\theta^2}{\gamma^3})t + \frac{\theta^2}{2\gamma^2}\ t^2]e^{-\mu t}$$

$$A_4(t) = [1 - \frac{\theta}{\gamma^2} + \frac{3\theta^2}{\gamma^4} - \frac{10\theta^3}{\gamma^6} + \frac{35\theta^4}{\gamma^8} + (\frac{\theta}{\gamma} - \frac{2\theta^2}{\gamma^3} + \tag{14}$$

$$+ \frac{6\theta^3}{\gamma^5} - \frac{20\theta^4}{\gamma^7})\ t + (\frac{\theta^2}{2\gamma^2} - \frac{3\theta^3}{2\gamma^4} + \frac{5\theta^4}{\gamma^6})t^2 +$$

$$+ (\frac{\theta^3}{6\gamma^3} - \frac{2\theta^4}{3\gamma^5})t^3 + \frac{\theta^4}{24\gamma^4}\ t^4] + \frac{\theta}{\gamma^2}\ [1 - \frac{3\theta}{\gamma^2}$$

$$+ \frac{10\theta^2}{\gamma^4} - \frac{35\theta^3}{\gamma^6} - (\frac{\theta}{\gamma^2} - \frac{4\theta^2}{\gamma^3} + \frac{15\theta^3}{\gamma^5})t$$

$$+ \frac{\theta^2}{2\gamma^2} - \frac{5\theta^3}{2\gamma^4})t^2 - \frac{\theta^3}{6\gamma^3}\ t^3]$$

The derivations of the exact expressions from equation (9) get tedious rapidly as k gets large. Consequently, an alternative approach was tried. Write (8) recursively for k = 1, 2,..., n to obtain:

$$\widetilde{A_1(s)} = \widetilde{\bar{F}(s)} + \widetilde{A_0(s)} \cdot \widetilde{H(s)} = \bar{F}(s)[1 + \widetilde{H(s)}] \tag{15-1}$$

$$\widetilde{A_2(s)} = \widetilde{\bar{F}(s)}\ [1 + \widetilde{H(s)} + \widetilde{H(s)}^2] \tag{15-2}$$

$$\widetilde{A_n(s)} = \widetilde{\bar{F}(s)}\ [1 + \widetilde{H(s)} + \widetilde{H(s)}^2 \tag{15-n}$$

$$+ \widetilde{H(s)}^3 + \cdots + \widetilde{H(s)}^n].$$

On inverting and simplifying algebraically we can determine the solution. For example, when $n \to \infty$, (15-n) becomes:

$$\lim_{n\to\infty} \widetilde{A_n(s)} = \widetilde{\bar{F}(s)} \left[\frac{1}{1-\widetilde{H(s)}}\right] \tag{16}$$

since $\widetilde{H(s)} < 1$ and a geometric series exists. Thus

$$\lim_{n\to\infty} \widetilde{A_n(s)} = \left(\frac{s}{\lambda s}\right)\left[\frac{1}{1-\widetilde{F(s)}\widetilde{G(s)}}\right] = \left(\frac{s}{\lambda s}\right)\left(\frac{1}{1-\frac{\lambda\mu}{(\lambda+s)(\mu+s)}}\right)$$

$$= 1 - \frac{\lambda}{s+(\lambda+\mu)} \tag{16a}$$

Inverting gives

$$\lim_{n\to\infty} A_n(t) = \int_0^t \delta(x)dx - \lambda \int_0^t e^{-(\mu+\lambda)x}dx$$

$$= \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t} \tag{17}$$

where $\delta(t)$ is the unit impulse function [4]. Taking the limit as $t \to \infty$ gives the steady state operational availability identical to Eq. (2) as it should be. Unfortunately, the inverse Laplace transforms of the expressions for $\widetilde{A_k(s)}$ are not in general easily obtained.

B. INCREMENTAL AVAILABILITY AND NORMAL APPROXIMATION

Now, we seek to determine expressions for the incremental benefit that can be obtained by adding an additional spare. Let $\Delta_k(t)$ be the incremental system availability that will result from the addition of one spare unit to a system already having k-1 spares. Then,

$$\Delta_k(t) = A_k(t) - A_{k-1}(t) \tag{18}$$

For small values of k, (18) can be evaluated from (9) by computing $A_k(t)$ and $A_{k-1}(t)$ and substracting. When k gets large (3 or more) the computations become tedious, as noted earlier. Therefore, some approximation method is needed to estimate (18) for large k. Using (15-n), we can write (18) in terms of its Laplace transforms as:

$$\widetilde{\Delta_k(s)} = \bar{F}(s) \cdot \overset{k}{H(s)} \tag{19}$$

and, inverting,

$$\Delta_k(t) = \int_0^t [1 - F(t-x)]dH_k(x) \tag{20}$$

$$= H_k(t) - \int_0^t F(t-x)dH_k(x)$$

However, note that $H_k(t) = P_r[T_1+R_1 + T_2+R_2 +\cdots+ T_k+R_k \leq t]$

and

$$\int_0^t F(t-x)dH_k(x) = P_r[T_1+R_1 + T_2+R_2+\cdots+T_k+R_k + T_{k+1} \leq t].$$

For large k, we can appeal to the Central Limit Theorem to use the normal approximation for the distribution of the sum of 2k random variables. Thus,

$$H_k(t) \approx N[k(\tfrac{1}{\lambda} + \tfrac{1}{\mu}),\ k(\tfrac{1}{\lambda^2} + \tfrac{1}{\mu^2})]$$

and

$$\int_0^t F(t-x)dH_k(x) \approx N[(k(\frac{1}{\lambda} + \frac{1}{\mu}) + \frac{1}{\lambda}),\ (k(\frac{1}{\lambda 2} + \frac{1}{\mu 2}) + \frac{1}{\lambda 2})].$$

Hence, for large k, (18) can be approximated by:

$$\Delta_k(t) \simeq \Delta_k^N(t) \doteq \Phi[\frac{t-k(\frac{1}{\lambda}+\frac{1}{\mu})}{\sigma_k}] - \Phi[\frac{t-(k(\frac{1}{\lambda}+\frac{1}{\mu})+\frac{1}{\lambda})}{\sigma_k'}] \qquad (21)$$

where $\sigma_k = \sqrt{k(\frac{1}{\lambda^2} + \frac{1}{\mu^2})}$ and $\sigma_k' = \sqrt{k(\frac{1}{\lambda^2} + \frac{1}{\mu^2}) + \frac{1}{\lambda^2}}$

Table 1 gives numerical expressions for $A_k(t)$, $\Delta_k(t)$ and $\Delta_k^N(t)$ for various values of k and the parameters MTBF = 30 days and MTTC = 5 days.

Table 1. Sample Calculations for $A_k(t)$, $\Delta_k(t)$ and $\Delta_k^N(t)$
(MTBF = 30 days, MTTC = 5 days)

t	45 Days			90 Days		
k	$A_k(t)$	$\Delta_k(t)$	$\Delta_k^N(t)$	$A_k(t)$	$\Delta_k(t)$	$\Delta_k^N(t)$
0	0.2231	N/A	N/A	0.0498	N/A	N/A
1	0.5712	0.3481	N/A	0.2171	0.1673	N/A
2	0.7784	0.2072	0.133	0.4622	0.2451	0.254
3	0.8429	0.0646	0.064	0.6721	0.2098	0.159
4	0.8554	0.0125	0.027	0.7909	0.1188	0.086
5			0.012			0.045
6			0.006			0.023
7			0.003			0.012
∞				0.8570	0.000	0.000

(1) N/A: Not Applicable (2) $\Delta_k^N(t)$ is normal approximation for $\Delta_k(t)$.

Figure 4 shows a plot of $A_k(t)$ for k = 0,1,2,3 and 4 for the parameters MTBF = 30 days and MTTC = 5 days. The curves show that little incremental benefit can be gained by increasing the number of spares beyond k=4.

The approach described here would appear to be fruitful. For small k, the expressions could be evaluated exactly and for k>4, the normal approximation could be utilized.

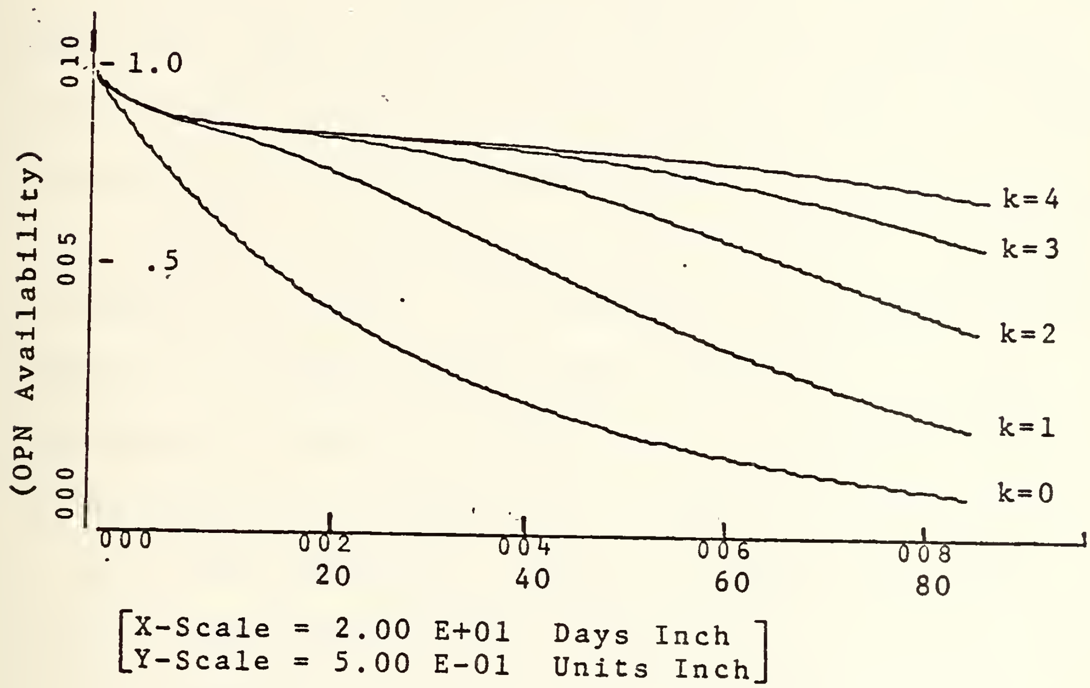


Figure 4. Operational Availability, $A_k(t)$.

III. OPERATIONAL AVAILABILITY FOR A SYSTEM WITH STANDBY SPARES AND A REPAIR CAPABILITY (TYPE-2 SYSTEM)

In addition to providing a larger number of standby spares, the system availability can be improved for a given spares allotment by increasing the equipment reliability or decreasing its replacement time or adding an onboard repair capability. This chapter investigates the latter alternative. The objective is to derive an expression for computing the operational availability of a system with standby spares with a repair capability. The tradeoff problem of adding more spares or adding a repair capability will be investigated.

Consider a system consisting of a single component that is subject to failure. The system is supported by a finite number k of standby spares which, when idle, do not deteriorate. When the system fails, a standby spare is instantaneously switched on (if a spare remains) and repair of the failed component commences immediately. (This implies that the replacement time is negligible.) We also assume that the repair facility has ample servers so that no failed unit has to wait for repairs to begin. (The repair facility is a (k+1) server queue.) Under these assumptions, the system (called a Type-2 System) is down only when the system fails and no standby units are available. Assume that the system fails and no standby units are available. Assume that the system and all standby spares are up at the start of the

mission. As before, assume that the times to failure for units in operation are exponential with parameter λ, and the times to repair are exponential with parameter β. Finally, assume that the times to failure and the repair times are independent.

Let $P_i(t)$ be probability that i units are in repair at time t for $i = 1,2,\ldots, k+1$. Then, the operational availability of a Type-2 System with k spares is

$$Q_k(t) = 1 - P_{k+1}(t). \tag{22}$$

For the special case where k=0, the system goes down and remains down for a repair interval every time the system fails. As soon as the repair is completed, the system becomes operational. Therefore, the process acts as an alternating renewal process, just as with the Type-1 System with $k=\infty$. By analogy, the operational availability is given by:

$$Q_o(t) = \frac{\beta}{\lambda+\beta} + \frac{\lambda}{\lambda+\beta} \exp[- t(\lambda+\beta)], \tag{23a}$$

and the steady-state operational availability is

$$\lim_{t\to\infty} Q_o(t) = \frac{\beta}{\lambda+\beta} . \tag{23b}$$

For the general case of k>0 spares, a different approach is taken. Assume first that k=1. Then, in a small interval of time Δt, the transitions described in Figure 4 can be made.

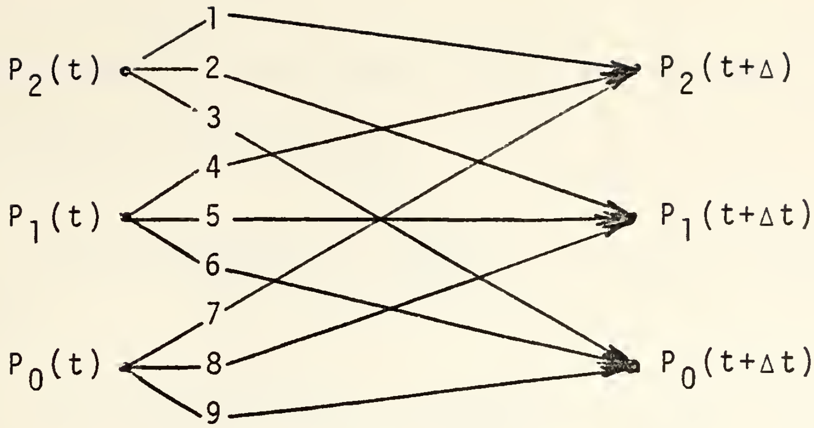


Figure 5. Transition Diagram (Type-2 System).

1 = P_r (no repair occurs) = $e^{-2\beta\Delta t} \approx 1-2\beta\Delta t$ [7]

2 = P_r (1 unit gets repaired) $\approx 2\beta\Delta t$

3 = P_r (2 units get repaired) $\approx \beta^2(\Delta t)^2 \approx 0$

4 = P_r (active unit fails and no units are repaired) $\approx \lambda\Delta t$

5 = P_r (active unit stays up and no units are repaired) + P_r (active unit fails and a failed unit gets repaired) = $1-(\lambda+\beta)\Delta t$

6 = P_r (active unit stays up and a failed unit gets repaired) $\approx \beta\Delta t$

7 = P_r (simultaneous failure of 2 units) ≈ 0

8 = P_r (active unit fails) $\approx \lambda\Delta t$

9 = P_r(active unit stays up) $\approx 1-\lambda\Delta t$.

Then, the following 3 simultaneous equations hold:

$$\begin{cases} P_0(t+\Delta t) = (1-\lambda\Delta t)\, P_0(t) + \beta\Delta t P_1(t) + 0 \cdot P_2(t) \\ P_1(t+\Delta t) = \lambda\Delta t\, P_0(t) + [1-(\lambda+\beta)\Delta t]\, P_1(t) + 2\beta\Delta t P_2(t) \\ P_2(t+\Delta t) = 0 \cdot P_0(t) + \lambda\Delta t P_1(t) + (1-2\beta\Delta t) P_2(t) \end{cases} \quad (24)$$

Rearranging (24) and taking the limit as $\Delta t \to \infty$, yields the 3 differential equations:

$$\begin{cases} \lim\limits_{\Delta t \to 0} \dfrac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = \dfrac{dP_0(t)}{dt} = -\lambda P_0(t) + \beta P_1(t) \\ \lim\limits_{\Delta t \to 0} \dfrac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = \dfrac{dP_1(t)}{dt} = \lambda P_0(t) - (\lambda+\beta) P_1(t) \\ \qquad + 2\beta P_2(t) \\ \lim\limits_{\Delta + \to 0} \dfrac{P_2(t+\Delta t) - P_2(t)}{\Delta t} = \dfrac{dP_2(t)}{dt} = \lambda P_1(t) - 2\beta P_2(t) \end{cases} \tag{25}$$

To solve (25), we take the Laplace transform of both sides of each equation and convert the transformed system of equations into a matrix representation. The above 2 steps are shown in (26) and (27) respectively.

$$\begin{cases} s\,\widetilde{P_0(s)} - 1 = -\lambda \widetilde{P_0}(s) + \beta \widetilde{P_1(s)} \\ s\,\widetilde{P_1(s)} = \widetilde{P_0}(s) - (\lambda+\beta)\widetilde{P_1}(s) + 2\beta \widetilde{P_2}(s) \\ s\,\widetilde{P_2(s)} = \lambda \widetilde{P_1}(s) - 2\beta \widetilde{P_2}(s) \end{cases} \tag{26}$$

Since $\mathcal{L}[\frac{dP_i(t)}{dt}] = s\,\widetilde{P_i(s)} - f(0^-)$ holds [6], and the initial value of $P_i(t)$ is equal to 1 in this problem (say, $P_0(0) = f(0^-) = 1$), (26) holds.

$$\begin{bmatrix} s+\lambda & -\beta & 0 \\ -\lambda & s+(\lambda+\beta) & -2\beta \\ 0 & -\lambda & s+2\beta \end{bmatrix} \begin{bmatrix} \widetilde{P_0}(s) \\ \widetilde{P_1}(s) \\ \widetilde{P_2}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

The solution for $P_2(t)$ is given by

$$P_2(t) = \frac{\lambda^2}{ab} \{1 + \frac{1}{(a-b)} [be^{-at} - ae^{-bt}]\}, \tag{28}$$

where $a = \frac{(3\beta+2\lambda) - \sqrt{\beta^2+4\lambda\beta}}{2}$

$b = \frac{3\beta+2\lambda + \sqrt{\beta^2+4\lambda\beta}}{2}$

Because $P_2(t)$ is the probability that the system will be "down" at time t for the case where only one spare is available, the operational availability of a Type-2 System for k=1 is

$$Q_1(t) = 1-P_2(t)$$

$$= 1- \frac{{}^2}{ab} \{1 + \frac{1}{(a-b)} [be^{-at}-ae^{-bt}]\}. \tag{29}$$

We are also interested in the asymtotic probability of (29), which is given by

$$\lim_{t\to\infty} Q_1(t) = \frac{2\beta(\lambda+\beta)}{2\beta^2+2\lambda\beta+\lambda^2} . \tag{30}$$

Analogous to the simple case above, for an arbitrary k, we have a (k+2) x (k+2) matrix system for solving $\widetilde{P}_{k+1}(s)$:

$$\begin{bmatrix} (s+\lambda) & -\beta & 0 & 0 - - & - - - - - - - - & 0 \\ -\lambda & (s+\lambda+\beta) & -2\beta & 0 - - & - - - - - - - - & 0 \\ 0 & -\lambda & (s+\lambda+2\beta) & -3\beta & - - - - - - - - & 0 \\ 0 & 0 & -\lambda & (s+\lambda+3\beta) & -4\beta - - - - - - - & 0 \\ 0 & 0 & 0 & -\lambda & (s+\lambda+4\beta) - - - - - - & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 - - - - - - - - & -\lambda \quad (s+\lambda+k\beta) & -(k+1)\beta \\ 0 & 0 & 0 & 0 - - - - - - - - - - - - & -\lambda & [s+\lambda+(kH)\beta] \end{bmatrix} \begin{bmatrix} \widetilde{P}_0(s) \\ \widetilde{P}_1(s) \\ \widetilde{P}_2(s) \\ \widetilde{P}_3(s) \\ \widetilde{P}_4(s) \\ \vdots \\ \widetilde{P}_k(s) \\ \widetilde{P}_{k+1}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

Solving (31) for $\widetilde{P}_{k+1}(s)$ gives:

$$P_{k+1}(s) = \frac{\lambda^{(k+1)}}{s[(s+a_1)(s+a_2) - - - (s+a_{k+1})]} \tag{32}$$

where $-a_1$, $-a_2$, - - - , $-a_{k+1}$ are the zeros of the polynomial in the denominator. All of the roots are functions of λ and β, and they can be found numerically by using various computer programs. When the roots are known, (32) can be expanded using partial fractions and then converted to find $P_{k+1}(t)$. This gives

$$P_{k+1}(s) = \frac{d_1}{s} - \frac{d_2}{s+a_1} - \frac{d_3}{s+a_2} - \ldots - \frac{d_{k+1}}{s+a_{k+1}} \tag{32a}$$

where $d_1, d_2, ---, d_{k+1}$ are all constants and

$$P_{k+1}(t) = d_1 e^{-a_1 t} - d_2 e^{-a_2 t} - d_3 e^{-a_3 t} - - - d_{k+1} e^{a_{k+1} t}.$$

Then, the general formula for computing the operational availability for a Type-2 System is

$$Q_k(t) = 1-(d_1 e^{-a_1 t} - d_2 e^{-a_2 t} - d_3 e^{-a_3 t} - - - d_{k+1} e^{-a_{k+1} t}). \tag{33}$$

For example, when k=2, the operational availability function is given by

$$Q_2(t) = 1-\lambda^3[\frac{1}{a_1 a_2 a_3} - \frac{1}{a_1(a_2-a_1)(a_3-a_1)} \exp(-a_1 t) - \frac{1}{a_2(a_1-a_2)(a_3-a_2)} \exp(-a_2 t) - \frac{1}{a_3(a_1-a_3)(a_2-a_3)} \exp(-a_3 t)] \tag{34}$$

Suppose that MTBF = 30 days, and MTTR = 5 days. Then,

$$P_3(s) = \frac{0.000037}{s(s+0.20033)(s+0.41097)(s+0.68870)},$$

and

$$P_3(t) = 0.000037 [\frac{1}{(0.20033)(0.41097)(0.68870)} - \frac{e^{-0.20033t}}{(0.20033)(0.41097-0.20033)(0.68870-0.20033)} - \frac{e^{-0.41097t}}{(0.41097)(0.20033-0.41097)(0.68870-0.41097)} - \frac{e^{-0.68870t}}{(0.68870)(0.20033-0.68870)(0.41097-0.68870)}]$$

and

$$Q_2(45 \text{ days}) = 1-P_3(45 \text{ days}) = 0.99935.$$

Various computed values of $Q_k(t)$ for k=0 through k=4 are tabulated in Table 2.

Table 2. Tabulated Values of Operational Availability for Type-2 System (MTBF = 30 days and MTTR = 5 days)

k \ t	45 days	90 days	∞
1	0.85715	0.85714	0.85714
2	0.98824	0.98823	0.98823
3	0.99935	0.99935	0.99935
4	0.99998	0.99998	0.99998
5	1.0	1.0	1.0

IV. REPAIR/SPARES TRADEOFF AND RESOURCE ALLOCATION ALGORITHMS

In the last two chapters we have shown that a system's operational availability can be improved by either increasing the number of spares dedicated to the support of the system, or by purchasing an onboard repair capability for the system. Both of these solutions involve difficult resource allocation decisions. In the case of adding more spares there is the problem of finding sufficient space to carry the support units. Also, because of tight procurement budgets, the various systems must compete for the funds. This creates a real resource allocation problem.

In the case of purchasing a repair capability there are perhaps even more difficult problems of manpower planning and training; purchasing and maintaining test equipment and repair equipment, and, also, the purchase and storage of spare stock. With the reduced manning levels that are being imposed on ship crews and with the sophistication of weapon systems, the recent trend has been away from onboard repair and more toward the removable module/replacement alternative. However, a tradeoff does exist and it may be that some systems should be Type-1 and others Type-2. In this chapter, we cannot quantify all of the costs associated with purchasing a repair capability, nor can we consider all of the actual resource constraints. Instead, we assume a simple cost structure for each type of system and look at the cost-benefit tradeoff decision.

Later in this chapter we consider the resource allocation problem. Given a repair/no repair decision has been made about each system and a fixed procurement budget, how should the dollars be spent among the many systems that require support? We also show the dramatic effect that the system configuration can have on the allocation procedure and on system availability.

A. DECISION RULE FOR REPAIR/SPARES TRADEOFF

Let us assume that the system consists of a single component that is subject to failure. Assume that the total expected repair cost for a mission of length t is linear, i.e.

$$C_f + C_r E[N(t)]$$

where N(t) is the random number of failures during the mission of length t, C_r is a cost (average) per unit of repair, and C_f is a fixed cost of repair that is independent of the number of units actually repaired. C_f would include the cost of training repair personnel, the cost of manning for repair, the cost of test equipment, etc. Further, assume that an operational availability goal for the mission is given. Then, our objective is to decide whether or not the repair capability should be purchased and what number of spares should be carried. The criterion is the minimization of costs. Stated as a nonlinear program:

$$\begin{cases} \min \text{ (cost)} \\ \text{s.t. } A_k(t)[\text{or } Q_k(t)] \geq p \end{cases} \qquad (35)$$

The spare stock procurement cost is simply $C_p \cdot k$ where C_p is the procurement cost per unit and k is the number of units stocked. The solution to (35) can be obtained as follows:

(1) Let k^* be the minimum k such that $A_k(t) \geq p$.

(2) Let k^{**} be the minimum k such that $Q_k(t) \geq p$.

(3) Compare $C_p \cdot k^*$ with $C_p \cdot k^{**} + \{C_f + C_r E[N(t)]\}$

where $N(t) \approx \lambda \int_0^t Q_{k^{**}}(t)dt$

(4) a. If $C_p \cdot k^* > C_p \cdot k^{**} + \{C_f + C_r E[N(t)]\}$, choose the Type-2 system with $k = k^{**}$.

b. If $C_p \cdot k^* < C_p \cdot k^{**} + \{C_f + C_r E[N(t)]\}$, the choose the Type-1 system with $k = k^*$.

c. If the costs are equal, select the maximum of $Q_{k^{**}}(t)$ and $A_{k^*}(t)$.

This decision problem is demonstrated with the following numerical example.

Suppose that p = 0.85, t = 90 days, MTBF = 50 days, MTTC = 1 day, MTTR = 5 days, C_p = \$400, C_f = \$500, and C_v = \$200 per repair.

<u>Step 1</u>. $A_3(90) = 0.8859 \rightarrow k^* = 3$

<u>Step 2</u>. $Q_0(90) = 0.9090 \rightarrow k^{**} = 0$

<u>Step 3</u>. $C_p \cdot k^* = \$1,500$

$$E[N(t)] \approx \lambda \int_0^t \left[\frac{\beta}{\lambda+\beta} + \frac{\lambda}{\lambda+\beta} e^{-(\lambda+\beta)X}\right] dx$$

$$E[N(90)] \approx 1.6363$$

$$C_p k^{**} + \{C_f + C_v E[N(90)]\} = \$727.4$$

Step 4. $C_p k^{*} > C_p k^{**} + \{C_f + C_v E[N(90)]\}$

$\therefore$ Type-2 System with k=0 is preferable.

B. DECISION RULE FOR THE OPTIMUM NUMBER OF SPARES FOR SUPPORT

The problem of realistically allocating spares for the support of equipment should take into consideration the "cost" of the system being down as well as the cost of spares and the cost of repairs. However, the problem of quantifying the "cost" of a system being down is very difficult. The difficulty lies in the fact that the cost is not so much in monetary units, but more in terms of reduced effectiveness where the word effectiveness is ill defined. Military planners have long agonized over the problems of measuring effectiveness and quantifying the cost of a system being down. We are still far from a solution today and the solution is beyond the scope of this paper. Nevertheless, as in the previous section, we assume a simple cost structure with known costs for the purpose of demonstrating a solution methodology.

Assume that the cost of the system being down is proportional to the down time and that the cost rate is known. Also, assume that the total expected system cost (TC) is the sum of the cost of spares, the expected cost of repairs, and the expected cost of the system down time. (For the Type-1 System the expected repair cost is omitted.) The cost of spares and the cost of repairs are as defined

in the previous section. Thus, the total expected cost of system downtime for the mission of length t is:

$$C_d \cdot E[D(t)]$$

where D(t) is the down time in (0,t), and the total expected system cost with k spares is:

$$TC(k) = C_p \cdot k + \{C_f + C_r \cdot E[N(t)]\} + C_d \cdot E[D(t)].$$

for a Type-2 system and

$$TC(k) = C_p \cdot k + C_d \cdot E[D(t)]$$

for a Type-1 system.

With this cost structure, the algorithm for determining the optimal number of spares for either of the two types of systems is:

(1) Let k=0

(2) If $TC(k+1) - TC(k) \geq 0$ set $k^* = k$ and stop.

(3) Set k=k+1 and go to (2).

Without additional examination of the costs as functions of k, no general claim for optimality can be made. However, actual experience has shown that such simple algorithms usually yield optimal or near optimal solutions. A numerical example of the procedure is given below. Suppose that C_d = \$100/day and that all the other costs and parameters are identical to those of the example in the previous section. We determine the number of spares, k^*, for both Type-1 and Type-2 Systems.

(a) Type-1 System:

$TC(1) - TC(0) = -\ 2155.79$

$TC(2) - TC91) = -\ 793.49$

$TC(3) - TC(2) = \quad 3.31 > 0 \rightarrow k^* = 2$

(b) Type-2 System:

$TC(1) - TC(0) = -\ 249.83$

$TC(2) - TC(1) = \quad 465.23 > 0 \rightarrow k^* = 1$

C. RESOURCE ALLOCATION SCHEME FOR INDIVIDUAL SYSTEM

Now, we consider the resource allocation problem. The budget available for supporting all the spares that are needed will not usually be sufficient to achieve the levels of operational availability that are specified for the systems. The budget constraints force us to consider effective allocation schemes through which a fixed number of dollars (B) can be allocated to the many systems being considered so as to maximize the operational availability per dollar spent on spares. This can be done by marginal analysis using benefit-to-cost tradeoffs. In this section, an algorithm for the selection of spares for support is developed. The algorithm considers the marginal increase in system operational availability per dollar spent. With such an algorithm a list of spares could be determined which represents the most economical way that supply can achieve its availability goal.

Suppose there are n independent Type-1 Systems. Define the following expressions:

(a) $P_{(i)}$ = the specified operational availability goal for the ith system.

(b) $C_{(i)}$ = the unit cost per spare associated with the ith system.

(c) $A_k^{(i)}(t) = A_k(t)$ for the ith system

(d) $\Delta_k^{(i)}(t) = \Delta_k(t)$ for the ith system.

Then, the recommended allocation algorithm is as follows:

<u>Step 1</u>. Compute $A_o^{(i)}(t)$ for i = 1,2,3, ... , n.
If $A_o^{(i)}(t) \geq P_{(i)}$ for every i, then no allocation of dollars is needed. Stop.
Otherwise, go to Step 2.

<u>Step 2</u>. If B < Min $\{C_{(i)}\}$ for ith system such that $A_o^{(i)}(t) < P_{(i)}$, then allocation is impossible. Stop.
If B = Min $\{C_{(i)}\}$ for ith system such that $A_o^{(i)}(t) < P_{(i)}$, then allocate $C_{(i)}$. Stop.
If B > Min $\{C_{(i)}\}$ for ith system such that $A_o^{(i)}(t) < P_{(i)}$, then go to Step 3.

Step 3. Compute Max $\{\frac{\Delta_1^{(i)}(t)}{C_{(i)}}\}$ (the greatest increase in system availability per dollar spent). Suppose the maximum occurs when i=j. If $B \geq C_{(j)}$, purchase one unit of system j. If $B < C_{(j)}$, select the largest

$\frac{\Delta_1^{(i)}(t)}{C_{(i)}}$ such that $C(i) \leq B$. If no units can be purchased, stop the procedure.

Step 4. If the item purchased in Step 3 is item j, next compute

$$\max \{\max_{i \neq j} [\frac{\Delta_1^{(i)}(t)}{C(i)}, \frac{\Delta_2^{(j)}(t)}{C(j)}\}$$

The next unit is assigned to the itme where the maximum is taken on (if the item can be purchased with the funds available). Continue this procedure until adding an additional unit would exceed the budget available.

Numerical Example. Suppose the budget available is B = $6,000, n = 4 systems, and the parameters are as given by Table 3.

Table 4 shows the order in which the allocation procedure selects items for support. The procedure stops with $5,700 spent since the remaining $300 cannot purchase any additional spares. The circled numbers indicate the order of selection. For example, the first item selected was item 2, then item 3, then item 1, etc.

Table 3. Parameters for Resource Allocation Example.

i	1	2	3	4
$C_{(i)}$	\$1,000	\$800	\$900	\$1,200
MTBF	30 days	40 days	50 days	30 days
MTTC	5 days	4 days	2 days	5 days
$P_{(i)}$	0.8	0.8	0.9	0.8

Table 4. Cost-Benefit Ratios and Selection Sequence

i	1	2	3	4
$\frac{\Delta_1(t)}{C(i)}$	(3) 0.000348	(1) 0.000457	(2) 0.000404	(4) 0.000290
$\frac{\Delta_2(t)}{C(i)}$	(6) 0.000207	(5) 0.000210	0.000164	0.000173
$\frac{\Delta_3(t)}{C(i)}$	0.000065	0.000053	0.000020	0.000054

The resource allocation procedure described above could easily be modified to reflect some sort of weighting scheme such as military essentiality codes if it is desired to consider the importance of the item in the allocation procedure. In that case, we would consider the ratios

$$\frac{E_i \Delta_k^{(i)}(t)}{C(i)}$$

where E_i is the item weighting factor. The procedure would then work just as described above. Again, no claim of optimality can be made for this marginal allocation procedure, but practical experience with such schemes shows that they usually produce good results.

V. THE AVAILABILITY PROBLEM WITH CONSIDERATION OF SYSTEM CONFIGURATION

In all of the preceding material we have assumed that the system consisted of a single component. This was done so that we could ignore the system configuration. In the real world problem, we would certainly not want to ignore system configuration, for the support levels of components of systems should depend on what happens to the system when the components fail. To determine this, one must examine the system configuration. The supply support procedure (COSAL) presently in use by the U.S. Naval Supply System totally ignores system configuration. In fact, it ignores the system itself, looking only at the components of those systems. Its contribution is that of piece part support. In this chapter we examine the impact that system configuration might have on operational availability and through an example we compare the allocations with the configuration considered and with the system configuration ignored.

Complex weapon systems usually consist of several major subsystems or components which are actually the items supported aboard ship. We therefore consider each system as broken down into its "supportable" components. We now view our objective as one of maximizing "system" availability. We seek to achieve our objective by supporting the components of that system. We confine our discussion to weapon systems

consisting of components arranged in series, parallel or simple combinations of both. We assume that each weapon system is a coherent system of order n, and its subsystems are Type-1 items. The analysis for Type-2 items is identical.

Let $A_{sys}(t)$ and P_{sys} be the operational availability of the weapon system, at time t and its specified availability goal, respectively. Let $\Delta_s^{(i)}(k,t)$ be the incremental system availability at time t obtained by adding one extra spare (from k-1 to k) of component i.

<u>Step 1</u>. Determine the system availability, $A_{sys}(t)$, from the component availability equations assuming zero spares for every component.

<u>Step 2</u>. Compute $\max_i \{\frac{\Delta_{sys}^{(i)}(1,t)}{C(i)}\}$. (The max is taken over all components in the system.) If the maximum is taken on for i=j and the budget is sufficient to purchase one unit then allocate C(j) dollars for component j. If not, select that component with the largest ratio that can be purchased with the available funds.

<u>Step 3</u>. Compute $\max \{\max_{i \neq j}[\frac{\Delta_{sys}^{(i)}(1,t)}{C(i)}], \frac{\Delta_{sys}^{(j)}(2,t)}{C(j)}\}$ and continue as in the allocation procedure of the last chapter until no additional components can be purchased.

<u>Step 4</u>. The shipboard allowance list for spare stock consists of the spares determined by this procedure.

Numerical Example. Consider the missile system of Figure 6 extracted from [7].

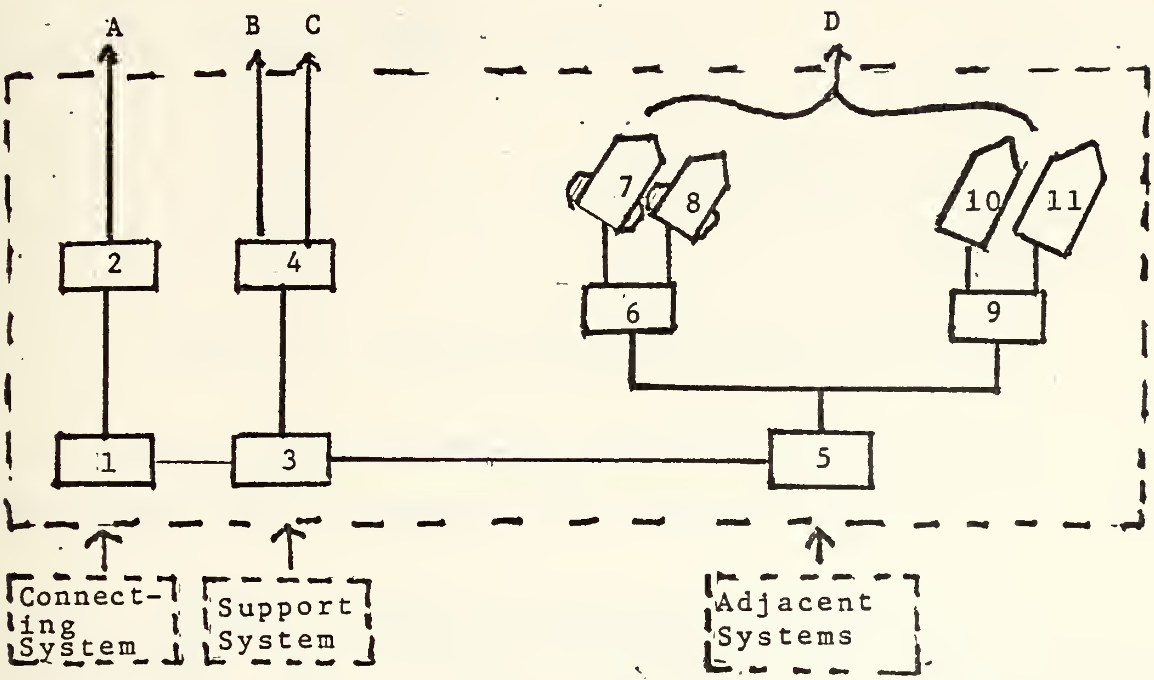


Figure 6. Weapon System Configuration.

Suppose that functions A, B, and C are necessary for a successful intercept, and that al least one missile must be launched. The reliability block diagram for system function D (target intercept) is shown in Figure 7.

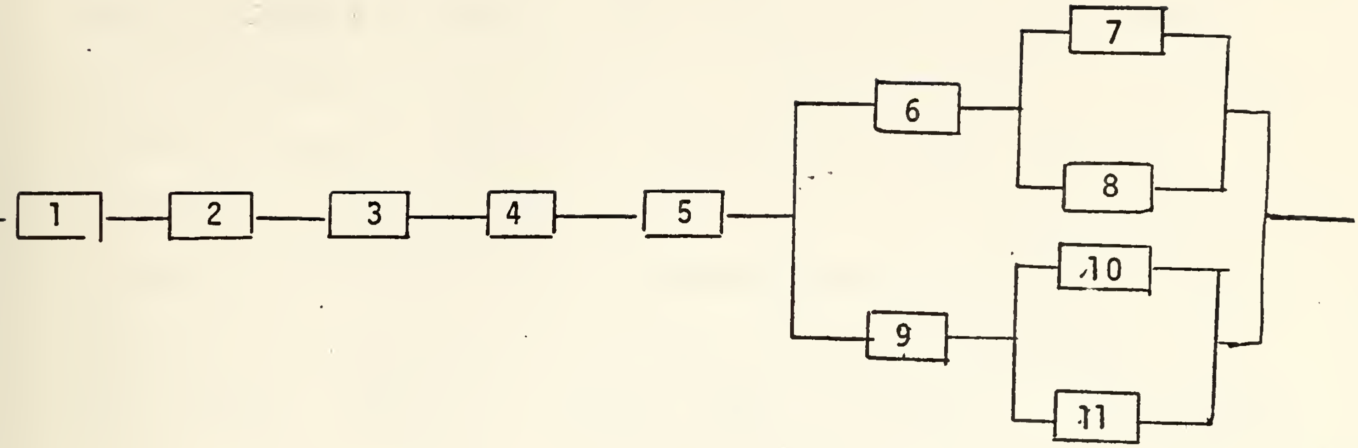


Figure 7. System Reliability Block Diagram.

Consider that the weapon system has 11 independent subsystems whose parameters are given in Table 5.

Table 5. Missile Subsystem Parameters

System (i)	MTBF (Days)	MTTC (Days)	Unit Price ($)	System (i)	MTBF (Days)	MTTC (Days)	Unit Price ($)
1	50	1	5,000	7	30	1	10,000
2	60	1	7,000	8	30	1	10,000
3	100	4	4,000	9	40	2	5,000
4	50	2	5,000	10	30	1	10,000
5	90	2	2,000	11	30	1	10,000
6	100	2	5,000				

Using series-parallel reliability calculations, the operational availability of the missile system is easily determined to be:

$$A_{sys}(t) = A_{k_1}^{(1)}(t) \cdot A_{k_2}^{(2)}(t) \cdot A_{k_3}^{(3)}(t) \cdot A_{K_4}^{(4)}(t) \cdot A_{k_5}^{(5)}(t) \cdot \{A_{k_6}^{(6)}(t) \cdot$$

$$[A_{k_7}^{(7)}(t) + A_{k_8}^{(8)}(t)] + A_{k_9}^{(9)}(t) \cdot [A_{k_{10}}^{(10)}(t) + A_{k_{11}}^{(11)}(t)]\}$$

For a system availability goal of P_{sys} = 0.80 and a budget of B = $76,000 and the component parameters given in Table 5, Table 6 shows the detailed computational steps with the allocation sequence indicated by the circled values. The column headings are the incremental benefit-to-cost ratios generated by adding an extra unit of the i^{th} component to the present listing of spares shown in column 1. The numbers in the vectors listed in column 1 denote the number of spares of each of the components. For example (2,1,0,1,0,0,0,1,0,0,0) means that there are presently 2 units of component 1, 1 unit of component 2, 1 unit of component 4 and 1 unit of component 8. In the example, the funds were used up before the availability goal was attained. The final spare parts determination for the missile system is (3,2,1,3,2,0,2,0,0,0,0) and the system availability at t = 45 is $A_{sys}(45)$ = 0.7097. For comparison we used the same example and determined the spare parts listing using the allocation scheme described in the last section of the previous chapter. Table 7 gives the detailed computational results for that procedure. As before, the numbers in circles represent the sequence of allocations. The final spare parts listing when the system configuration is ignored is given by (2,1,0,2,2,0,1,1,1,1,1). With this allowance list the system availability is given by

$$A_{sys}(45) = 0.5183$$

Table 6. Alocation Procedure for Missile System

(For actual $\frac{\Delta_{sys}^{(i)}(t)}{C_{(i)}}$ values, divide by 100,000) (t = 45 days)

	$\frac{\Delta_s^{(1)}(t)}{C_{(1)}}$	$\frac{\Delta_s^{(2)}(t)}{C_{(2)}}$	$\frac{\Delta_s^{(3)}(t)}{C_{(3)}}$	$\frac{\Delta_s^{(4)}(t)}{C_{(4)}}$	$\frac{\Delta_s^{(5)}(t)}{C_{(5)}}$	$\frac{\Delta_s^{(6)}(t)}{C_{(6)}}$	$\frac{\Delta_s^{(7)}(t)}{C_{(7)}}$	$\frac{\Delta_s^{(8)}(t)}{C_{(8)}}$	$\frac{\Delta_s^{(9)}(t)}{C_{(9)}}$	$\frac{\Delta_s^{(10)}(t)}{C_{(10)}}$	$\frac{\Delta_s^{(11)}(t)}{C_{(11)}}$	Money Spent
(0,0,0,0,0,0,0,0,0,0,0)	0.37	0.22	0.02	0.37	0.51	0.01	0.35	0.35	0.08	0.35	0.35	$C_{(5)}$: $2000
(0,0,0,0,1,0,0,0,0,0,0)	0.56	0.33	0.28	0.55	0.11	0.02	0.25	0.25	0.12	0.25	0.25	$C_{(1)}$: $5,000
(1,0,0,0,1,0,0,0,0,0,0)	0.24	0.62	0.06	1.05	0.21	0.04	0.28	0.28	0.23	0.28	0.28	$C_{(4)}$: $5,000
(1,0,0,1,1,0,0,0,0,0,0)	0.45	1.18	0.11	0.43	0.41	0.07	0.53*	0.53*	0.44	0.53*	0.53*	$C_{(2)}$: $7,000
(1,1,0,1,1,0,0,0,0,0,0)	0.79	0.42	0.20	0.75	0.71	0.12	0.93	0.93	0.76	0.93	0.93	$C_{(7)}$: $10,000
(1,1,0,1,1,0,1,0,0,0,0)	1.16	0.62	0.29	1.10	1.05	0.20	0.67	0.52	0.45	0.13	0.13	$C_{(1)}$: $5,000
(2,1,0,1,1,0,1,0,0,0,0)	0.32	0.75	0.35	1.32	1.27	0.24	0.81	0.63	0.55	0.16	0.16	$C_{(4)}$: $5,000
(2,1,0,2,1,0,1,0,0,0,0)	0.38	0.89	0.42	0.33	1.52	0.29	0.96	0.75	0.65	0.19	0.19	$C_{(5)}$: $2,000
(2,1,0,2,2,0,1,0,0,0,0)	0.22	0.95	0.45	0.35	0.18	0.30	1.03	0.80	0.69	0.19	0.19	$C_{(7)}$: $10,000
(2,1,0,2,2,0,2,0,0,0,0)	0.50	1.18	0.56	0.43	0.24	0.40	0.22	0.36	0.36	0.10	0.10	$C_{(2)}$: $10,000
(2,2,0,2,2,0,2,0,0,0,0)	0.58	0.27	0.64	0.50	0.28	0.46	0.26	0.41	0.41	0.12	0.12	$C_{(1)}$: $4,000
(2,2,1,2,2,0,2,0,0,0,0)	0.60	0.28	0.01	0.52	0.29	0.48	0.27	0.43	0.43	0.13	0.13	$C_{(1)}$: $5,000
(3,2,1,2,2,0,2,0,0,0,0)	0.12	0.29	0.01	0.54	0.30	0.50	0.28	0.45	0.45	0.13	0.13	$C_{(4)}$: $5,000
(3,2,1,3,2,0,2,0,0,0,0)												Total: $75,000

*Indifferent purchase among 7th, 8th, 10th and 11th systems.

Table 7. Allocation Procedure for Missile System (Component Oriented).

(For actual $\frac{\Delta_k(t)}{C_{(i)}}$ values, divide by 10,000) (t = 45 days)

Component	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th
$\frac{\Delta_1(t)}{C_{(i)}}$	(3) 0.7298	(5) 0.5030	0.0983	(4) 0.7270	(1) 1.480	0.0824	(5) 0.3383	(7) 0.3383	(2) 0.7328	(8) 0.3383	(9) 0.3383
$\frac{\Delta_2(t)}{C_{(i)}}$	(11) 0.3128	0.1791	0.0018	(12) 0.2958	(10) 0.3315	0.0016	0.2448	0.2448	0.1371	0.2448	0.2448
$\frac{\Delta_3(t)}{C_{(i)}}$	0.0854	0.0406	0.00002	0.0730	0.0425	0.00002	0.0532	0.0532	0.1171	0.0532	0.0532

We note that this availability is far below what was obtained when the system configuration was explicitly considered even though the latter procedure spent $1000 more. This is only one example out of many possibilities, but it does point out the importance of considering the system configuration in spare parts allocation schemes. We cannot claim that the improvement would always be as dramatic as that witnessed in this example but there must always be some improvement. After all, the objective is to maximize system availability, not the component availabilities. Therefore, it only makes sense to focus on the system and not the components.

VI. SUMMARY AND CONCLUSIONS

In this chapter summary and conclusions are made concerning the discussions and results presented in this report. Following the conclusions, suggestions are given for extension and enrichment of this study.

We have derived a mathematical expression to compute the operational availabilities for a standby spares system having no repair capability when there are finitely many spare parts for support. By employing a Markovian approach which, in turn, permits us to work with linear homogeneous differential equations with constant coefficients, we have determined a formula for calculating the operational availability for a standby spares system with a repair capability. We have discussed decision rules for determining if a repair capability should be purchased, a decision rule for determining the number of spares for the two systems, and finally resource allocation schemes for components within a system. We have shown that the resource allocation scheme should consider the system configuration when making decisions about those items that should be supported. Certainly, if the attainment of a specified level of system availability is the objective, then resources should be allocated to support those components which are most critical, not necessarily those with the highest rates of failure. In our example, we witnessed a potential money savings of about 45% considering system configuration as compared to the one which neglects

the system configuration. These results support the argument that the Supply System should become more maintenance and system oriented and less piece parts oriented.

In describing the operational availability of a given system (non-maintained system) it is necessary to specify (1) the equipment failure process, (2) the replacement mechanism, (3) the system configuration, and (4) the state in which system is to be defined as failed. For maintained systems, we should also consider the repair mechanism. The simplest hypothesis from a mathematical viewpoint is to assume that the equipment (component) failure processes, replacement processes and repair processes have exponential distributions. These assumptions make it relatively easy to determine the operational availability of a system as a function of the availabilities of its components when there are limited numbersof spare parts for support. These operational availability functions, in turn, allow us to compute the correct operational availability of a complex system by considering its system configuration. As demonstrated in Chapters IV and V, these functions can then be used as decision criteria for the determination of spare part listings for support, such as the COSAL. When the dollars available for constructing a shipboard parts list are not sufficient to achieve specified levels of operational availability for all systems, the resource allocation scheme should consider the marginal

increase in system operational availability per dollar invested in order to maximize the system operational availability. It is also emphasized that system configuration should be considered throughout the resource allocation process. Without these considerations, the limited resources might be spent in an inefficient way or perhaps additional funds would be expended to achieve the same level of operational availability which could otherwise have been obtained less expensively.

In modeling the operational availability functions, exponential distributions have been assumed not only for the failure process, but also for the replacement and repair processes. This might not be the case in many applications. Since, in many cases, the use of the exponential failure law can be justified, this might not be critical for the component lifetimes. However, for the replacement and repair mechanisms, the assumption is probably unrealistic. Relaxation of the assumption results in complicated expressions for operational availability. Even with the simplest model shown in (9), it is very laborious to compute $A_k(t)$ values, when k gets large. This suggests the importance of developing good approximation methods. Even though this paper tries to solve basic issues associated with the shortcomings of current COSAL procedure, many problems such as the definition of operational availability, military essentiality, system configuration, availability goals, and mission scenarios still require much

work before many significant changes can be made to the COSAL procedure. It is recommended that further study along these lines be conducted.

It is hoped that this study will be helpful to those people who endeavor to improve the current COSAL procedure of the Naval Supply Systems Command.

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