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## On digit sums of multiples of an integer

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## ABSTRACT

Let  $g > 1$  be an integer and  $s_g(m)$  be the sum of digits in base  $g$  of the positive integer  $m$ . In this paper, we study the positive integers  $n$  such that  $s_g(n)$  and  $s_g(kn)$  satisfy certain relations for a fixed, or arbitrary positive integer  $k$ . In the first part of the paper, we prove that if  $n$  is not a power of  $g$ , then there exists a nontrivial multiple of  $n$  say  $kn$  such that  $s_g(n) = s_g(kn)$ . In the second part of the paper, we show that for any  $K > 0$  the set of the integers  $n$  satisfying  $s_g(n) \leq Ks_g(kn)$  for all  $k \in \mathbb{N}$  is of asymptotic density 0. This gives an affirmative answer to a question of W.M. Schmidt.

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## 1. Introduction

The distribution of various arithmetic functions on positive integers written in given base  $g > 1$  is a classical area of investigation and a huge body of literature on this topic has been published. In this paper, we denote by  $s_g(n)$  the sum of digits in base  $g$  of the positive integer  $n$  and study positive integers  $n$  such that there is some relation between  $s_g(n)$  and  $s_g(kn)$  for some small positive integer  $k$  which might be specified or not. For example, first we study the numbers  $n$  having a nontrivial multiple  $m$  (that is,  $m/n$  is not a power of  $g$ ) such that  $s_g(n) = s_g(m)$ . More precisely, let

$$\mathcal{N}_g = \{n: s_g(n) = s_g(kn) \text{ for some integer } k \neq g^\ell\}.$$

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Note that  $\mathcal{N}_g$  misses some integers. For example, no power of  $g$  belongs to  $\mathcal{N}_g$ . Our first result is that  $\mathcal{N}_g$  contains all positive integers except for the powers of  $g$ .

**Theorem 1.** *The set  $\mathcal{N}_g$  consists of all positive integers which are not a power of  $g$ .*

Let  $(a_g(n))_{n \geq 1}$  be the sequence defined by

$$a_g(n) = \min\{kn : k \neq g^\ell, s_g(n) = s_g(kn)\}, \quad \text{if } n \in \mathcal{N}_g,$$

and  $a_g(n) = 0$ , otherwise. Our Theorem 1 shows that  $a_g(n) \neq 0$  for all  $n \neq g^\ell$ . In base 10, we have

$$(a_{10}(n))_{n \geq 1} = (0, 110, 12, 112, 140, 24, 133, 152, 18, 0, 1001, 300, 2002, \dots).$$

This is sequence A087303 in [10].

It seems interesting to study the extremal orders of  $a_g(n)$ . Our next result gives some nontrivial estimates on  $a_g(n)$ .

**Theorem 2.** *The inequalities*

$$n \left( \frac{\log n}{\log \log n} \right)^{1/3} \ll a_g(n) \leq \exp \left( \frac{n}{(\log n)^{(1+o(1)) \log \log \log n}} \right)$$

hold for almost all positive integers  $n$ .

Taking  $n = g^m + 1$ , we see easily that  $a_g(n) = g^{3m} + 1$ . In particular,  $a_g(n) \asymp n^3$  holds for infinitely many  $n$ . Furthermore, taking  $n = 1 + g + \dots + g^{p-1}$  where  $p$  is a prime number and noting that  $M = 1 + g^2 + \dots + g^{p-1} + g^{p+1} = n - g + g^{p+1}$  is a nontrivial multiple of  $n$  with the same sum of digits, we get that in fact  $a_g(n) \asymp n$  also holds for infinitely many  $n$ .

A problem closely related to the determination of the set  $\mathcal{N}_g$  is to study the set of positive integers  $n$  such that the inequality  $s_g(n) \leq s_g(kn)$  holds with some fixed integer  $k \geq 2$  coprime with  $g$ . In 1980, Stolarsky [12] proved that the set of positive integers  $n$  satisfying the above inequality with  $g = 2$  and  $k = 3$  has density  $1/2$ . He called such numbers *3-sturdy* numbers. He also introduced the *k-sturdy* numbers as being the numbers  $n$  such that  $s_2(n) \leq s_2(kn)$ , as well as the *sturdy* numbers, as being the numbers  $n$  which are *k-sturdy* for all positive integers  $k$ .

Schmidt [9] proved that for every fixed odd integer  $k \geq 3$ , the set of *k-sturdy* numbers has density  $1/2$ . He also showed that the sturdy numbers have density 0.

He asked (see p. 608 in [9]) for the following generalization of this problem.

**Question 3.** *Given  $K > 0$ , is it true that the set of numbers  $n$  with*

$$s_2(n) \leq K s_2(kn) \quad \text{for all } k \geq 1,$$

*forms a set of asymptotic density 0?*

In this paper, we give an affirmative answer to Schmidt’s question.

**Theorem 4.** *Let  $g \geq 2$  and  $K > 0$ . For  $x \geq 3$ , the number of positive integers  $n \leq x$  satisfying*

$$s_g(n) \leq K s_g(kn) \quad \text{for all } k \geq 1,$$

*is  $O(x/(\log x)^{1/2})$ . In particular, such numbers form a set of asymptotic density 0.*

The above result is interesting only for  $K \geq 1$ , since for  $K \in (0, 1)$  it is a direct consequence of Theorem 1. With a bit more care, it is possible to replace the exponent  $1/2$  on the logarithm in Theorem 4 above by  $1 - \varepsilon$ , where  $\varepsilon > 0$  is arbitrarily small.

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1, which is elementary. Section 3 consists of the proof of Theorem 2, which uses some results on the joint distribution of  $s_g(n)$  and  $s_g(hn)$  in arithmetic progressions with small prime moduli, due to the first author and G. Tenenbaum [4], as well as an upper bound for the Carmichael  $\lambda$ -function of  $n$  valid for almost all positive integers  $n$ , due to P. Erdős, C. Pomerance and E. Schmutz [5]. The last section is devoted to the proof of Theorem 4. The idea of this proof is to apply a recent result of Bourgain [2] on exponential sums. For this, we also need an estimate of Tenenbaum [14] on the number of positive integers  $n \leq x$  with a large divisor having only small prime factors, a result of Banks, Garaev, and the second author [1] on the order of the subgroup of  $(\mathbb{Z}/n\mathbb{Z})^*$  generated by the number  $g$ , and an upper bound on the counting function of the positive integers  $n$  such that  $s_g(n)$  is abnormally small obtained by Mauduit, Pomerance and Sárközy [6].

Throughout this paper, we use the Landau symbols  $O$  and  $o$  and the Vinogradov symbols  $\gg, \ll$  and  $\asymp$  with their usual meanings. The constants implied by them depend on the number  $g$ . We denote by  $P^+(n)$  (respectively by  $P^-(n)$ ) the greatest (respectively the smallest) prime factor of the positive integer  $n$  with the conventions  $P^+(1) = 1, P^-(1) = \infty$ . We use  $p$  and  $q$  with or without subscripts for prime numbers.

**2. The proof of Theorem 1**

First suppose that  $n$  has at least two nonzero digits in base  $g$ :

$$n = c_1g^{\ell_1} + \dots + c_tg^{\ell_t},$$

with  $t \geq 2, 1 \leq c_1, \dots, c_t \leq g - 1$  and  $0 \leq \ell_1 < \dots < \ell_t$ . Write  $n = ab$ , where all the primes dividing  $a$  divide also  $g$  and  $b$  is coprime to  $g$ . Clearly, there exists a natural number  $\alpha$  such that  $a \mid g^\alpha$ . Since  $(b, g) = 1$ , it follows that  $g^{\lambda(b)} \equiv 1 \pmod{b}$ , where for a positive integer  $m$  we use  $\lambda(m)$  for the Carmichael function of  $m$ , which is the maximal order of invertible elements modulo  $m$ . Thus,  $g^{\ell_t + \alpha + \lambda(b)} \equiv g^{\ell_t + \alpha} \pmod{n}$ . We now put

$$M = c_1g^{\ell_1 + \alpha} + \dots + c_{t-1}g^{\ell_{t-1} + \alpha} + c_tg^{\ell_t + \lambda(b) + \alpha}.$$

We then have

$$M \equiv c_1g^{\ell_1 + \alpha} + \dots + c_tg^{\ell_t + \alpha} \pmod{n} \equiv ng^\alpha \pmod{n} \equiv 0 \pmod{n}.$$

It is also obvious that  $s_g(M) = s_g(n)$ . Since  $g^{\ell_1} \parallel n$  and  $g^{\alpha + \ell_1} \parallel M$ , the equality  $M = g^k n$  can only be satisfied by the integer  $k = \alpha$ . But we easily check that  $M - g^\alpha n = c_t g^{\ell_t + \alpha} (g^{\lambda(b)} - 1) \neq 0$ . Thus,  $M/n$  is not a power of  $g$ .

Suppose next that  $n$  has only one nonzero digit in base  $g$ . If  $n = g^\ell$ , then it is clear that  $n \notin \mathcal{N}_g$ . Thus, assume now that  $n = cg^\ell$ , where  $2 \leq c \leq g - 1$ . If  $c$  and  $g$  are coprime, we then take  $M = g^\ell(c - 1 + g^{\lambda(c)})$  and it is clear that  $n \mid M$  and that  $M/n = 1 + (g^{\lambda(c)} - 1)/c$  is not a power of  $g$ . More generally, if  $c = c_1c_2$ , where  $c_1 \geq 2$  is coprime to  $g$  and all prime factors of  $c_2$  divide  $g$ , we then take  $\alpha$  to be such that  $c_2 \mid g^\alpha$  and put  $M = g^{\ell + \alpha}(c_1c_2 - 1 + g^{\lambda(c_1)})$ . The integer  $M$  has two nonzero digits and  $n$  only one, thus  $M/n$  is not a power of  $g$ .

Finally, if  $n = cg^\ell$ , where all prime factors of  $c$  divide  $g$ , we can then take  $M = g^{\ell + \alpha}(c - 1 + g)$ , where again  $\alpha$  is sufficiently large such that  $c \mid g^\alpha$ .

### 3. The proof of Theorem 2

If we write  $n = ab$ , where all prime factors of  $a$  divide  $g$  and  $b$  is coprime to  $n$ , then for almost all positive integers  $n$  we have that  $a < \log n$ . Now the proof of Theorem 1 shows that the inequality

$$a_g(n) \leq g^{\alpha(n)+\lambda(n)} n \tag{1}$$

holds for all positive integers  $n \in \mathcal{N}_g$ , where we put  $\alpha(n)$  for the smallest  $\alpha$  such that  $a \mid g^\alpha$ . Since  $a < \log n$  for almost  $n$ , it follows easily that  $g^{\alpha(n)} \ll (\log n)^{\omega(g)}$ , where  $\omega(g)$  is the number of distinct prime factors of  $g$ . In particular, the inequality  $g^{\alpha(n)} < n$  holds for all sufficiently large  $n$  in a set of asymptotic density 1.

Now Theorem 2 of [5] shows that the inequality

$$\lambda(n) \leq n \exp\left(-\left(1 + o(1)\right) \log \log n \log \log \log n\right)$$

holds as  $n \rightarrow \infty$  in a set of asymptotic density 1. Thus, using inequality (1), one gets easily that

$$a_g(n) \leq \exp\left(\frac{n}{(\log n)^{(1+o(1)) \log \log \log n}}\right)$$

holds as  $n \rightarrow \infty$  in a set of asymptotic density 1, which is the upper bound of Theorem 2.

For the lower bound, let  $x$  be large and let  $H$  be some parameter depending on  $x$  and tending to infinity with  $x$  to be chosen later. We count the number of  $n \leq x$  such that  $s_g(n) = s_g(hn)$  holds with some positive integer  $2 \leq h \leq H$  which is coprime to  $g$ . Let us fix the number  $H$ . Let  $p > \max\{H(\log H)^{1/2}, g\}$  be some prime depending also on  $x$  to be fixed later. Corollary 2.10 in [4] shows that under these conditions, we have

$$\begin{aligned} \mathcal{A}_{h,p,a}(x) &= \#\{n \leq x: s_g(n) \equiv s_g(hn) \equiv a \pmod{p}\} \\ &= \frac{x}{p^2} + O\left(x^{1-c_0/(p^2 H \log H)}\right), \end{aligned}$$

where  $c_0$  is a positive constant depending on  $g$ . Assuming that

$$x^{c_0/(p^2 H \log H)} > p^3, \tag{2}$$

and that  $x$  is large, we then see that the inequality

$$\mathcal{A}_{h,p,a}(x) < \frac{2x}{p^2}$$

holds uniformly for  $2 \leq h \leq H$  and  $a \in \{0, 1, \dots, p - 1\}$ . Thus,

$$\begin{aligned} \mathcal{A}(x) &= \sum_{\substack{2 \leq h \leq H \\ g \nmid h}} \#\{n \leq x: s_g(n) = s_g(hn)\} \\ &\leq \sum_{\substack{2 \leq h \leq H \\ g \nmid h}} \#\{n \leq x: s_g(n) = s_g(hn) \pmod{p}\} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{2 \leq h \leq H \\ g|h}} \sum_{0 \leq a \leq p-1} \mathcal{A}_{h,p,a}(x) \\
 &\leq \frac{2Hx}{p} \ll \frac{x}{(\log H)^{1/2}} = o(x)
 \end{aligned} \tag{3}$$

as  $x \rightarrow \infty$ . Inequality (2) tells us that we should choose  $p$  such that

$$c_0 \log x > 3p^2 H \log H \log p.$$

Choosing  $p$  to be the first prime  $> H \log H$ , we get that the above estimate implies that

$$c_0 \log x \geq (3 + o(1))(H \log H)^3,$$

giving

$$H = (c_1 + o(1)) \left( \frac{\log x}{\log_2 x} \right)^{1/3} \text{ as } x \rightarrow \infty,$$

where  $c_1 = (9c_0)^{1/3}$ . Since,  $a_g(n) \geq Hn$  holds for all  $n \leq x$  with the exception of a set of positive integers  $n$  of cardinality  $o(x)$  as  $x \rightarrow \infty$  (see inequality (3)), the lower bound follows. The proof of Theorem 2 is complete.

Schmid [8] proved that if  $k_1 \neq k_2$  are fixed odd integers and  $a$  is a fixed integer, then the asymptotic

$$\#\{n: 0 \leq n < x: s_2(k_1 n) - s_2(k_2 n) = a\} \sim \frac{x}{\sqrt{2\pi V} (\log x / \log 2)} e^{-\frac{a^2 \log 2}{2V \log x}}$$

holds as  $x \rightarrow \infty$ , where

$$V = \frac{1}{2} \left( 1 - \frac{\gcd(k_1, k_2)^2}{k_1 k_2} \right).$$

Applying this asymptotic with  $(k_1, k_2) = (1, 3)$  yields the lower bound

$$\#\{n \leq x: a_2(n) = 3n\} \gg \frac{x}{\sqrt{\log x}}.$$

It is perhaps possible to generalize the result of Schmid to other bases. Steiner [11] obtained a very general result on the global joint distribution of sequences  $\{s_g(P_\ell(n))\}_{n \geq 1}$ ,  $\ell = 1, \dots, d$ , where  $P_1, \dots, P_\ell$  are polynomials in  $\mathbb{Z}[X]$ . It would be very interesting to obtain a local result for the above joint distribution at least for the case of linear polynomials.

#### 4. The proof of Theorem 4

For coprime positive integers  $g$  and  $n$ , let  $t_g(n)$  denote the order of  $g$  modulo  $n$ . For  $x \geq 2$ , we define the following functions of  $x$ :

$$\begin{aligned}
 y_0 &= \exp((\log x)^{1/2}), & z_0 &= \exp((\log x)^{3/4}), \\
 y_1 &= \exp((\log x)^{1/8}), & z_1 &= \exp((\log x)^{1/4}).
 \end{aligned} \tag{4}$$

Every positive integer  $n$  has a decomposition of the form  $n = a_n b_n$  with  $P^+(a_n) \leq y_0 < P^-(b_n)$ .

The idea of this proof is the following. Since  $a_n$  is small for almost integers  $n$ , its sum of digits is very small. The main ingredient of the proof then consists in finding for almost all positive integers  $n$ , a multiple of  $b_n$ , say  $kb_n$ , with very few and sparse nonzero digits, such that  $a_n(kb_n) = kn$  has a small sum of digits.

Define the following sets of primes

$$\mathcal{P} = \{p: t_g(p) < p^{1/2}(\log p)^{-10}\}$$

and

$$\mathcal{Q}(x) = \{p \leq x: p \equiv 1 \pmod{d} \text{ for some } d > z_1 \text{ with } P^+(d) < y_1\}.$$

We denote by  $\mathcal{P}(x)$  the set  $\mathcal{P} \cap [1, x]$ . The following lemma turns out to be quite useful.

**Lemma 5.** *The following estimates hold:*

$$\#\mathcal{P}(x) = O\left(\frac{x}{(\log x)^2}\right), \tag{5}$$

$$\#\mathcal{Q}(x) \ll x \exp(-c_0(\log x)^{1/8}(\log \log x)), \tag{6}$$

where  $c_0 = 1/16$ . In particular,

$$\sum_{\substack{p > y \\ p \in \mathcal{P}}} \frac{1}{p} \ll \frac{1}{\log y}, \tag{7}$$

and

$$\sum_{\substack{p > y \\ p \in \mathcal{Q}(x)}} \frac{1}{p} \ll \exp(-c_1(\log y)^{1/8}(\log \log y)), \tag{8}$$

where  $c_1 = 1/17$ .

**Proof.** Estimate (5) appears in Lemma 4 in [1], and estimates (7) and (8) follow from estimates (5) and (6) by Abel's summation formula. Thus, it remains to prove estimate (6). For  $x \geq 1, y \geq 1, z \geq 1$ , we define the function  $\Theta(x, y, z)$  by

$$\Theta(x, y, z) = \#\left\{n \leq x: \prod_{\substack{p^v \parallel n \\ p \leq y}} p^v > z\right\}. \tag{9}$$

Let  $u = \log x / \log y, v = \log z / \log y$  and  $\varrho$  be the Dickman-de Bruijn function. Tenenbaum [14, Eq. (1.5)] showed that for all  $\varepsilon > 0$  the asymptotic

$$\Theta(x, y, z) \sim e^{-\gamma} x \int_v^\infty \varrho(t) dt$$

holds for

$$y \geq 2, \quad 1 \leq z \leq \exp \exp \{ (\log y)^{3/5-\varepsilon} \}, \quad z \leq x^{1-\varepsilon},$$

provided that both  $u$  and  $y$  tend to infinity. More precise asymptotic formulas for  $\Theta(x, y, z)$  appear in [13] and [14] (see also [3] for a less precise asymptotic which is valid in a larger range for  $z$  versus  $y$ ).

We now have

$$\mathcal{Q}(x) \subset \left\{ p \leq x: \prod_{\substack{q^v \parallel p-1 \\ q \leq y_1}} q^v > z_1 \right\}.$$

Since the function  $\varrho$  satisfies  $\varrho(t) = t^{-t+o(t)}$  for  $t \rightarrow \infty$ , we have

$$\#\mathcal{Q}(x) \leq \Theta(x, y_1, z_1) \ll x e^{-v(\log v)/2} \ll x \exp(-c_0(\log x)^{1/8}(\log \log x)), \tag{10}$$

which is what we wanted.  $\square$

Before continuing, we point out that the cardinality of the set  $\mathcal{Q}(x)$  with a somewhat larger value of  $y_1$  was also studied in Theorem 2 in [7].

**Lemma 6.** *Let  $x \geq 2$ . The following properties are satisfied by almost all integers  $n \leq x$  with at most  $O(x/(\log x)^{1/2})$  exceptions:*

- (i)  $a_n \leq z_0$ ;
- (ii)  $b_n$  is squarefree;
- (iii) If  $p \mid b_n$ , then  $p \notin \mathcal{P} \cup \mathcal{Q}(x)$ , where these sets of primes are defined in the beginning of this section;
- (iv) If  $d \mid b_n$ , then  $t_g(d) > d^{2/5}$ .

**Proof.** We assume that  $n > x/(\log x)^{1/2}$ , since there are at most  $x/(\log x)^{1/2}$  positive integers  $n$  failing this condition.

We begin with (i). By the definition of  $\Theta$  given in (9), the number of positive integers  $n \leq x$  not satisfying (i) is exactly  $\Theta(x, y_0, z_0)$ . As in estimate (10), we have

$$\Theta(x, y_0, z_0) \ll x e^{-v(\log v)/2} \ll x \exp(-c_2(\log x)^{1/4}(\log \log x)),$$

where  $c_2 = 1/8$ . Thus, there are at most  $O(x/(\log x)^{1/2})$  positive integers  $n \leq x$  failing (i).

If  $n \leq x$  fails (ii), then there exists a prime  $p > y_0$  such that  $p^2 \mid n$ . Given  $p$ , the number  $n \leq x$  which are multiples of  $p^2$  is  $\leq x/p^2$ . Thus, the number of  $n \leq x$  failing (ii) is at most

$$\leq \sum_{y_0 \leq p \leq x^{1/2}} \frac{x}{p^2} \ll \frac{x}{y_0} = \frac{x}{(\log x)^{1/2}}. \tag{11}$$

If  $n \leq x$  fails (iii), then there exists  $p \in \mathcal{P} \cup \mathcal{Q}(x)$  such that  $p \mid n$  and  $p > y_0$ . The number of such positive integers  $n \leq x$  is at most

$$\sum_{\substack{p \in \mathcal{P} \\ p > y_0}} \frac{x}{p} + \sum_{\substack{p \in \mathcal{Q}(x) \\ p > y_0}} \frac{x}{p} \ll \frac{x}{\log y_0} = \frac{x}{(\log x)^{1/2}}.$$



It remains to deal with the positive integers  $n \leq x$  failing (iv). We may assume that such  $n$  satisfies (i), (ii) and (iii), and let  $d$  be a divisor of  $b_n$ . Then  $d$  is also squarefree, and we have

$$t_g(d) = [t_g(p), p \mid d],$$

where  $[t_g(p), p \mid d]$  denotes the least common multiple of all the numbers  $t_g(p)$  taken over all the prime factors  $p$  of  $d$ . Since  $t_g(p) \mid p - 1$  holds for all  $p$ , it follows that if  $q \mid \gcd(t_g(p), t_g(p'))$  for two distinct prime factors  $p$  and  $p'$  of  $d$ , then  $q \mid \gcd(p - 1, p' - 1)$ . The number of positive integers  $n \leq x$  for which there exists  $q > y_1$  such that  $q \mid \gcd(t_g(p), t_g(p'))$  for two distinct primes factors  $p, p'$  of  $b_n$  is at most

$$\sum_{\substack{y_1 < q \leq \sqrt{x} \\ p, p' \equiv 1 \pmod{q}}} \frac{x}{pp'} \leq x \sum_{q > y_1} \left( \sum_{\substack{m \leq x \\ m \equiv 1 \pmod{q}}} \frac{1}{m} \right)^2 \ll x(\log x)^2 \sum_{q > y_1} \frac{1}{q^2} \ll \frac{x(\log x)^2}{y_1}.$$

For the remaining  $n$ , we have

$$[t_g(p), p \mid d] \geq \prod_{p \mid d} \frac{t_g(p)}{\prod_{p \mid d} \prod_{\substack{q^v \parallel p-1 \\ q \leq y_1}} q^v}.$$

We now use the fact that, by (iii), the prime factors of  $d$  do not belong to  $\mathcal{P} \cup \mathcal{Q}(x)$ . Thus,

$$[t_g(p), p \mid d] \geq \frac{\sqrt{d}}{(z_1(\log d)^{10})^{\omega(d)}},$$

where  $\omega(d)$  stands for the number of distinct prime factors of  $n$ . By the well-known Hardy–Ramanujan bounds,  $\omega(n)$  is very close to  $\log \log n$  for almost all  $n$ . More precisely,

$$\begin{aligned} \#\{n \leq x: \omega(n) > 10 \log \log x\} &\leq \sum_{n \leq x} 3^{\omega(n) - 10 \log \log x} \\ &\ll x(\log x)^{-10 \log 3} \prod_{p \leq x} (1 + 3/p) \\ &\ll x(\log x)^{3 - 10 \log 3} \ll x(\log x)^{-7}. \end{aligned}$$

Thus, for all positive integers  $n \leq x$  with at most  $O(x/(\log x)^{1/2})$  exceptions, we have

$$[t_g(p), p \mid d] \geq \frac{\sqrt{d}}{(z_1(\log d)^{10})^{10 \log \log x}}.$$

We now recall that  $P^-(d) > y_0$ . Thus, for  $x$  large enough we have

$$z_1^{10 \log \log x} = \exp(10(\log x)^{1/4} \log \log x) \leq y_0^{1/20} \leq d^{1/20}$$

and

$$(\log d)^{100 \log \log x} \leq \exp(100(\log \log x)^2) \leq d^{1/20}.$$

Thus, (iv) is also satisfied for all positive integers  $n \leq x$  with at most  $O(x/(\log x)^{1/2})$  exceptions.  $\square$

We prove the following lemma.

**Lemma 7.** *Let  $n$  be an integer satisfying Lemma 6. There exists an absolute constant  $\ell_0 \in \mathbb{N}$  such that for  $\ell \geq \ell_0$  and  $x \geq x_0(\ell_0)$  large enough, the congruence*

$$g^{j_1} + \dots + g^{j_\ell} \equiv 0 \pmod{b_n} \tag{12}$$

has a solution  $(j_1, \dots, j_\ell) \in \{0, \dots, t_g(b_n) - 1\}^\ell$ .

**Proof.** For  $t \in \mathbb{R}$ , we use the notation  $\mathbf{e}(t) = \exp(2i\pi t)$ . Let  $S_\ell$  denotes the number of solutions in  $\{0, \dots, t_g(b_n) - 1\}^\ell$  to the congruence (12). We have

$$\begin{aligned} S_\ell &= \frac{1}{b_n} \sum_{h=1}^{b_n} \left( \sum_{j=0}^{t_g(b_n)-1} \mathbf{e}\left(\frac{hg^j}{b_n}\right) \right)^\ell \\ &= \frac{(t_g(b_n))^\ell}{b_n} + \frac{1}{b_n} \sum_{h=1}^{b_n-1} \left( \sum_{j=0}^{t_g(b_n)-1} \mathbf{e}\left(\frac{hg^j}{b_n}\right) \right)^\ell \\ &= \frac{(t_g(b_n))^\ell}{b_n} + R_\ell, \end{aligned} \tag{13}$$

say. We will prove that for  $\ell$  large enough the term  $R_\ell$  is sufficiently small. The main tool for this proof is the following lemma of Bourgain [2].

**Lemma 8.** *Let  $q \in \mathbb{N}$  and  $H \subset (\mathbb{Z}/q\mathbb{Z})^*$  be a multiplicative subgroup such that  $|H| > q^\delta$ ,  $\delta > 0$  arbitrary. Then one has the exponential sum estimate*

$$\max_{a \in \mathbb{Z}_q^*} \left| \sum_{x \in H} \mathbf{e}_q(ax) \right| < c_3 q^{-\varepsilon} |H| \quad \text{for some } c_3 > 0 \text{ and } \varepsilon = \varepsilon(\delta) > 0.$$

To apply the above lemma, we must work with irreducible fractions  $h/d$ . We write

$$R_\ell = \frac{1}{b_n} \sum_{\substack{d|b_n \\ d \neq 1}} \sum_{\substack{1 \leq h \leq d \\ \gcd(h, b_n)=1}} \left( \sum_{j=0}^{t_g(b_n)-1} \mathbf{e}\left(\frac{hg^j}{d}\right) \right)^\ell.$$

If we write  $j = u + \lambda t_g(d)$  with  $0 \leq u < t_g(d)$  and  $0 \leq \lambda < t_g(b_n)/t_g(d)$ , we then have

$$g^j \equiv g^u \pmod{d}.$$

Thus, we also have

$$|R_\ell| \leq \frac{1}{b_n} \sum_{\substack{d|b_n \\ d \neq 1}} \sum_{\substack{1 \leq h \leq d \\ \gcd(h, b_n)=1}} \left( \frac{t_g(b_n)}{t_g(d)} \right)^\ell \left| \sum_{j=0}^{t_g(d)-1} \mathbf{e}\left(\frac{hg^j}{d}\right) \right|^\ell.$$

By Lemma 6, we have that  $t_g(d) > d^{2/5}$  holds for all  $d \mid b_n$  with  $d > 1$ . Thus, for each  $d$  we may apply Lemma 8 with  $H = \langle g \rangle$ , which is the subgroup of  $(\mathbb{Z}/d\mathbb{Z})^*$  generated by  $g$ , and with  $\delta = 2/5$ . We obtain the following estimate for  $R_\ell$ :

$$|R_\ell| \leq \frac{(c_3 t_g(b_n))^\ell}{b_n} \sum_{\substack{d \mid b_n \\ d \neq 1}} \varphi(d) d^{-\ell \varepsilon} \leq \frac{(c_3 t_g(b_n))^\ell}{b_n} \frac{\pi^2}{6} y_0^{3-\ell \varepsilon}. \tag{14}$$

Here,  $\varepsilon$  and  $c_3$  are the positive constants given by Lemma 8. In the last upper bound above, we used the fact that the inequality  $\varphi(d) d^{-\ell \varepsilon} \leq d^{-2} y_0^{3-\ell \varepsilon}$  holds for all  $d > y_0$  when  $3 - \ell \varepsilon < 0$ . Finally, combining estimates (13) with the bound (14) on  $R_\ell$ , we have

$$S_\ell \geq \frac{t_g(b_n)^\ell}{b_n} \left( 1 - \frac{\pi^2}{6} c_3^\ell y_0^{3-\ell \varepsilon} \right) > 0,$$

provided that  $\ell > 3\varepsilon^{-1}$  and that  $x$  is large enough, which completes the proof of Lemma 7.  $\square$

We are now ready to prove Theorem 4.

Let  $K > 0$ . Let  $x$  be large enough. Let  $\mathcal{M}(x)$  be the set of positive integers  $n \leq x$  satisfying the conditions of Lemma 6 such that

$$\left| s_g(n) - \frac{(g-1)}{2} \left\lfloor \frac{\log x}{\log g} \right\rfloor \right| \leq \frac{(g-1)}{4} \left\lfloor \frac{\log x}{\log g} \right\rfloor. \tag{15}$$

Lemma 4 in [6] shows that the number of positive integers  $n \leq x$  not satisfying estimate (15) is

$$\ll x^{1-c_4/\log \log x}, \tag{16}$$

where  $c_4$  is a positive constant depending on  $g$ . Thus, we certainly have

$$\#\mathcal{M}(x) = x + O(x(\log x)^{-1/2}).$$

Let  $n \in \mathcal{M}(x)$ . We will show that there exists a multiple  $kn$  of it such that  $s_g(kn) \leq K s_g(n)$ .

By Lemma 6,  $a_n \leq z_0$ . Thus,  $s_g(a_n) \leq g \log z_0 / \log g = g(\log x)^{3/4} / \log g$  and  $a_n < g^T$ , where  $T = 1 + \lfloor \log z_0 / \log g \rfloor$ . Next, we choose a fixed integer  $\ell \geq \ell_0$ , where  $\ell_0$  is defined by Lemma 7. By this lemma, there exists  $(j_1, \dots, j_\ell) \in \{0, \dots, t_g(b_n) - 1\}^\ell$  such that

$$\sum_{m=1}^{\ell} g^{j_m} \equiv 0 \pmod{b_n}.$$

Since  $g^{t_g(b_n)} \equiv 1 \pmod{b_n}$ , we also have

$$U = \sum_{m=1}^{\ell} g^{j_m + mT t_g(b_n)} \equiv 0 \pmod{b_n},$$

and the exponents of the nonzero digits of  $U$  are distinct. Furthermore,  $a_n U$  is a multiple of  $n$  such that

$$s_g(a_n U) = s_g(a_n) \ell \ll (\log x)^{3/4} < K^{-1} s_g(n)$$

for  $x$  large enough.

Indeed writing  $a_n = \sum_{i=0}^{T-1} c_i g^i$  with  $0 \leq c_i \leq g-1$ , we have

$$a_n U = \sum_{\substack{0 \leq i \leq T-1 \\ 1 \leq m \leq \ell}} c_i g^{i+j_m+mTt_g(b_n)} \quad (17)$$

and

$$i + j_m + mTt_g(b_n) = i' + j_{m'} + m'Tt_g(b_n) \Rightarrow m = m' \quad \text{and so} \quad i = i'.$$

Hence, the exponents in (17) are distinct. This completes the proof of Theorem 4.

As we have already mentioned, it is possible to replace the upper bound  $x(\log x)^{-1/2}$  on the cardinality of the set of exceptional positive integers  $n \leq x$  by  $x(\log x)^{-\alpha_0}$  for any  $\alpha_0 \in (0, 1)$  by taking

$$\begin{aligned} y_0 &= \exp((\log x)^{\alpha_0}), & z_0 &= \exp((\log x)^{\beta_0}), \\ y_1 &= \exp((\log x)^{\alpha_1}), & z_1 &= \exp((\log x)^{\beta_1}), \end{aligned}$$

with  $0 < \alpha_1 < \beta_1 < \alpha_0 < \beta_0 < 1$ . We do not enter into such details.

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## References

- [1] W.D. Banks, M.Z. Garaev, F. Luca, I.E. Shparlinski, Uniform distribution of fractional parts related to pseudoprimes, *Canad. J. Math.*, in press.
- [2] J. Bourgain, Exponential sum estimates over subgroups of  $\mathbb{Z}_q^*$ ,  $q$  arbitrary, *J. Anal. Math.* 97 (2005) 317–355.
- [3] E.R. Canfield, P. Erdős, C. Pomerance, On a problem of Oppenheim concerning 'Factorisatio Numerorum', *J. Number Theory* 17 (1983) 1–28.
- [4] C. Dartyge, G. Tenenbaum, Sommes des chiffres de multiples d'entiers, *Ann. Inst. Fourier* 7 (2005) 2423–2474.
- [5] P. Erdős, C. Pomerance, E. Schmutz, Carmichael's lambda function, *Acta Arith.* 58 (1991) 363–385.
- [6] C. Mauduit, C. Pomerance, A. Sárközy, On the distribution in residue classes of integers with a fixed digit sum, *Ramanujan J.* 9 (2005) 45–62.
- [7] C. Pomerance, I.E. Shparlinski, Smooth orders and cryptographic applications, in: *Proc. ANTS-V, Sydney, Australia*, in: *Lecture Notes in Comput. Sci.*, vol. 2369, Springer, 2002, pp. 338–348.
- [8] J. Schmid, The joint distribution of the binary digits of integer multiples, *Acta Arith.* 43 (1984) 391–415.
- [9] W.M. Schmidt, The joint distribution of the digits of certain integer  $s$ -tuples, *Stud. Pure Math.*, Birkhäuser, Basel, 1983.
- [10] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, <http://www.research.att.com/~njas/sequences/>.
- [11] W. Steiner, On the joint distribution of  $q$ -additive functions on polynomial sequences, in: *Proceedings of the Conference Dedicated to the 90th Anniversary of Boris Vladimirovich Gnedenko, Kyiv, 2002*, *Theory Stoch. Process.* 8 (2002) 336–357.
- [12] K.B. Stolarsky, Integers whose multiples have anomalous digital frequencies, *Acta Arith.* 38 (1980) 117–128.
- [13] G. Tenenbaum, Crible d'Ératosthène et modèle de Kubilius, in: K. Györy, H. Iwaniec, J. Urbanowicz (Eds.), *Number Theory in Progress, Proceedings of the Conference in Honor of Andrzej Schinzel, Zakopane, Poland, 1997*, Walter de Gruyter, Berlin, 1999, pp. 1099–1129.
- [14] G. Tenenbaum, Integers with a large friable component, *Acta Arith.* 124 (2006) 287–291.