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SUM OF DIVISORS OF FIBONACCI NUMBERS

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ABSTRACT. In this note, we prove an estimate on the count of Fibonacci numbers whose sum of divisors is also a Fibonacci number. As a corollary, we find that the series of reciprocals of indices of such Fibonacci numbers is convergent.

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1. Introduction

For a positive integer \( n \), we write \( \sigma(n) \) for the sum of divisors function of \( n \). Recall that a number \( n \) is called {\em multiply perfect} if \( n \mid \sigma(n) \). If \( \sigma(n) = 2n \), then \( n \) is called {\em perfect}. Let \((F_n)_{n \geq 1}\) be the sequence of Fibonacci numbers. In [4], it was shown that there are only finitely many multiply perfect Fibonacci numbers, and in [5], it was shown that no Fibonacci number is perfect. For a positive integer \( n \), the value \( \phi(n) \) of the Euler function is defined to be the number of natural numbers less than or equal to \( n \) and coprime to \( n \). In [6], it was shown that if \( \phi(F_n) = F_m \) then \( n \in \{1, 2, 3, 4\} \).

In [7], Fibonacci numbers \( F_n \) with the property that the sum of their aliquot parts is also a Fibonacci number were investigated. This reduces to studying those positive integers \( n \) such that \( \sigma(F_n) = F_n + F_m \) holds with some positive integers \( m \). In [7], it was shown that such positive integers form a set of asymptotic density zero.

Here, we search for Fibonacci numbers \( F_n \) such that \( \sigma(F_n) \) is a Fibonacci number. We put

\[ A = \{ n : \sigma(F_n) = F_m \text{ for some positive integer } m \} . \]

For a positive real number \( x \) and a subset \( B \) of the positive integers, we write \( B(x) = B \cap [1, x] \). In this note, we prove the following result.

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Theorem 1. There are constants $c_0$ and $C_0$ such that inequality

$$\#A(x) < \frac{C_0 x \log \log \log x}{(\log x)^2}$$

holds for all $x > c_0$.

By partial summation, Theorem 1 immediately implies that

Corollary 1.1. The series

$$\sum_{n \in A} \frac{1}{n}$$

is convergent.

We remark that it is quite possible that $A \setminus \{1, 2, 3\}$ is empty, as computer searches for $n \leq 5 \cdot 10^3$ failed to find any other element of $A$. The presumably larger set $B = \{n : \sigma(n) = F_m \text{ for some positive integer } m\}$ contains the integers $1, 2, 7, 9, 66, 70, 94, 115, 119, 2479$. It is likely that $B$ is infinite, but this is probably hard to prove.

Throughout this paper, we use the Vinogradov symbols $\gg, \ll$ and the Landau symbols $O, \asymp$ with their usual meanings. We recall that $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent and mean that $|A| < cB$ holds with some constant $c$, while $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold. For a positive real number $x$ we write $\log x$ for the maximum between 1 and the natural logarithm of $x$. We use $p, q, P$ and $Q$ to denote prime numbers.

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2. The Proof

Let $x$ be a large positive real number and assume that $n \leq x$. We also assume that $n > x/(\log x)^2$, since there are at most $x/(\log x)^2$ positive integers failing this property.
2.1. The size of \( m \) in terms of \( n \)

It is known that \( \sigma(n)/n \ll \log \log n \) (see Theorem 323 in Chapter 18 of [3]). Let \( \gamma = (1 + \sqrt{5})/2 \) be the golden section. Since \( F_n \approx \gamma^n \), we get that

\[
\gamma^{m-n} \ll \frac{F_m}{F_n} = \frac{\sigma(F_n)}{F_n} \ll \log \log F_n \ll \log n \leq \log x,
\]

therefore

\[
m - n < c_1 \log \log x
\]

holds for all sufficiently large values of \( x \), where we can take \( c_1 = 3 \). From now on, we write \( m = n + k \), where \( k < K = \lfloor c_1 \log \log x \rfloor \).

2.2. Discarding smooth integers

Let \( P(n) \) be the largest prime factor of \( n \). Let

\[
y = \exp \left( \frac{\log x \log \log x}{3 \log \log x} \right).
\]

Let

\[
\mathcal{A}_1(x) = \{ n \leq x : P(n) \leq y \}.
\]

The elements of the set \( \mathcal{A}_1(x) \) are referred to as \( y \)-smooth numbers. By known results from the distribution of smooth numbers (see, for example, Chapter III.5 from [8]),

\[
\#\mathcal{A}_1(x) \leq x \exp\left( -(1 + o(1))u \log u \right),
\]

where \( u = \log x/\log y \). In our case, we have \( u = 3 \log \log x/\log \log x \), therefore \( u \log u = 3(1 + o(1)) \log \log x \), leading to

\[
\#\mathcal{A}_1(x) \leq \frac{x}{(\log x)^{1+o(1)}} < \frac{x}{(\log x)^2},
\]

once \( x \) is sufficiently large.

2.3. The order of apparition of \( \sigma(F_{P(n)}) \)

For every positive integer \( n \) we write \( z(n) \) for the order of apparition of \( n \) in the Fibonacci sequence which is defined as the smallest positive integer \( u \) such that \( n \mid F_u \). It is known [2] that if \( n \mid F_t \), then \( z(n) \mid t \), and that \( z(n) \gg \log n \).

Let \( n \leq x \) be not in \( \mathcal{A}_1(x) \). Let \( p = P(n) \) be its largest prime factor. Then \( F_p \mid F_n \). We now show that \( F_p \) and \( F_n/F_p \) are coprime. It is known [1, Prop. 2.1] that

\[
\gcd\left( F_p, \frac{F_n}{F_p} \right) \mid \frac{n}{p}
\]

If the greatest common divisor appearing above were not 1, then there would exist a prime \( Q \mid F_p \) such that \( Q \mid n/p \). However, for large \( y \) (hence, for
large $x$), $Q \equiv \pm 1 \pmod{p}$, therefore $Q \geq 2p - 1 > p$, and so it cannot divide $n/p$ which is a $p$-smooth number. Thus, $F_p$ and $F_n/F_p$ are coprime, and by the multiplicative property of $\sigma$ we get that $\sigma(F_p) \mid \sigma(F_n)$. Hence, $\sigma(F_p) \mid F_m$, leading to $z(\sigma(F_p)) \mid m$.

Fix $p$ and $k = m - n$. Then $p \mid n$ and $z(\sigma(F_p)) \mid n + k$. Further, note that $p$ cannot divide $z(\sigma(F_p))$, for if this were the case, then the above congruences would lead to $p \mid k$, which is impossible for large $x$ since $0 < k \leq K < y < p$. Thus, we can apply the Chinese Remainder Lemma and conclude that $n$ is in a certain arithmetic progression modulo $pz(\sigma(F_p))$. Let $n_{k,p}$ be the least positive term of this progression, and let $A_{k,p}(x) = \{ n_{k,p} + pz(\sigma(F_p)) \lambda : \lambda > 0 \} \cap [1, x]$.

It is clear that $\#A_{k,p}(x) \leq [x/pz(\sigma(F_p))] \leq x/pz(\sigma(F_p))$, therefore if we write $A_2(x) = \bigcup_{0 < k \leq K} \bigcup_{y \leq p \leq x} A_{k,p}(x)$, (3)

then we have the bound

$$\#ar{A}_2(x) \leq \sum_{0 < k \leq K} \sum_{y \leq p \leq x} \frac{x}{pz(\sigma(F_p))} \ll xK \sum_{y \leq p \leq x} \frac{1}{y^2} \ll \frac{x \log \log x}{y},$$ (4)

where in the above estimate we used the fact that $z(\sigma(F_p)) \gg \log(\sigma(F_p)) \geq \log(F_p) \gg p$.

We put $A_3(x) = \{ n_{k,p} : k \in [1, K] \text{ and } p \in [y, x] \}$ (5)

and study $A_3(x)$. Let $L_1 = (\log x)^2$, $L = (\log x)/2$ put $z_1 = x/L_1$, $z = x/L$, and write

$$A_3(x) = A_4(x) \cup A_5(x) \cup A_6(x),$$

where

$$A_4(x) = A_3(x) \cap \{ n \leq x : P(n) < z_1 \},$$

$$A_5(x) = A_3(x) \cap \{ n \leq x : z_1 \leq P(n) < z \},$$

$$A_6(x) = A_3(x) \cap \{ n \leq x : z \leq P(n) \}.$$

Since elements of $A_4(x)$ are uniquely determined by their largest prime factor (at most $z_1$) and $k \in [1, K]$, we get that

$$\#A_4(x) \leq K \pi(z_1) \leq \frac{x(\log \log x)^2}{(\log x)^3},$$ (6)
once $x$ is sufficiently large. We will show that
\[ \#\mathcal{A}_5(x) \ll x \frac{\log \log x}{(\log x)^2} \]  
and that $\mathcal{A}_6(x)$ is empty for large values of $x$ which, together with estimates (2), (4) and (6), will complete the proof of the theorem.

2.4. The end of the proof

From now on until the end of the proof, $n$ is a positive integer in $\mathcal{A}_5(x) \cup \mathcal{A}_6(x)$. Then $n = pa$, where $a \leq L_1$. Thus, $F_a \mid F_n$. Put $A = F_n/F_a$ and note that every prime factor $P$ of $A$ has the property that $p \mid z(P)$. In what follows, we will estimate $\sigma(A)/A$. First of all
\[ \sigma(A)/A \leq \frac{A}{\varphi(A)} = \prod_{P \mid A} \left(1 + \frac{1}{P - 1}\right) \leq \prod_{d \mid a} \prod_{z(P) = pd} \left(1 + \frac{1}{P - 1}\right). \] 
(8)

For each fixed $d \mid a$, we have
\[ \prod_{z(P) = pd} \left(1 + \frac{1}{P - 1}\right) \leq \exp \left(\sum_{z(P) = pd} \frac{1}{P - 1}\right). \]

It is known (see, for example, [7]), that for each fixed positive integer $t$ we have
\[ \sum_{z(P) = t} \frac{1}{P - 1} \ll \frac{\log t}{\varphi(t)}. \]

Hence,
\[ \prod_{z(P) = pd} \left(1 + \frac{1}{P - 1}\right) \leq \exp \left(O \left(\frac{\log \log(pd)}{p\varphi(d)}\right)\right) = \exp \left(O \left(\frac{\log x}{p\varphi(d)}\right)\right). \] 
(9)

Thus, multiplying estimates (9) over all the divisors $d$ of $a$ and using (8), we get
\[ 1 \leq \frac{\sigma(A)}{A} \leq \exp \left(O \left(\frac{\log x}{p} \sum_{d \mid a} \frac{1}{\varphi(d)}\right)\right) \ll \exp \left(\frac{(\log x)^2}{p}\right) \]
for large $x$, where we used the fact that
\[ \sum_{d \mid a} \frac{1}{\varphi(d)} \ll \log \log a \sum_{d \mid a} \frac{1}{d} \leq \frac{\sigma(a) \log \log L_1}{a} \ll (\log \log L_1)^2 = o(\log \log x) \]
as $x \to \infty$. Hence,
\[ 0 < \frac{\sigma(A)}{A} - 1 < \exp \left(\frac{(\log \log x)^2}{p}\right) - 1 \leq \frac{2(\log \log x)^2}{p} \leq \frac{2(\log \log x)^2}{z_1}, \] 
(10)
where in the last inequality we used the fact that
\[
\frac{(\log \log x)^2}{z_1} \leq \frac{(\log \log x)^2}{z_1} = o(1)
\]
as \(x \to \infty\) together with the fact that the inequality \(e^t - 1 < 2t\) holds for all sufficiently small positive values of \(t\).

We will use that \(\sigma(F_n)/F_n\) is close to \(\sigma(F_a)/F_a\) since
\[
\sigma(F_a)/F_a < \sigma(F_n)/F_n \leq \frac{\sigma(F_a) \sigma(A)}{A}. \tag{11}
\]
In particular,
\[
\sigma(F_n)/F_n \ll \sigma(F_a)/F_a.
\]
Therefore,
\[
k = m - n \ll \log \left( \frac{\sigma(F_n)}{F_n} \right) \ll \log \left( \frac{\sigma(F_a)}{F_a} \right) \ll \log \log \log x.
\]

Now we are ready to estimate \(\#A_5(x)\):
\[
\#A_5(x) \ll \pi(L) \log \log \log x.
\]
This completes the proof of (7).

We now turn to the study of \(A_6(x)\). We have to show that \(A_6(x) = \emptyset\). Assume that \(n \in A_6(x)\). By (11),
\[
\frac{\sigma(A)}{A} - 1 \geq \frac{F_m}{A \sigma(F_a)} - 1 = \frac{F_m F_a}{F_n \sigma(F_a)} - 1.
\]

Writing \(F_t = (\gamma^t - \delta^t)/(\gamma - \delta)\), where \(\delta = (1 - \sqrt{5})/2 = -1/\gamma\), we get easily that
\[
\frac{F_m F_a}{F_n \sigma(F_a)} - 1 = \frac{\gamma^{m-n} F_a - \sigma(F_a)}{\sigma(F_a)} + O(\gamma^{-2n}). \tag{12}
\]
Since \(\gamma\) is quadratic irrational, it follows that the inequality
\[
|U\gamma - V| > \frac{c_3}{U}
\]
holds for all positive integers \(U\) and \(V\) with some positive constant \(c_3\). Since \(\gamma^{m-n} = F_{m-n} + F_{m-n-1}\), it follows that
\[
|\gamma^{m-n} F_a - \sigma(F_a)| = \left| \left( F_{m-n} F_a \right) (\gamma - (\sigma(F_a) - F_a F_{m-n+1})) \right|
\]
\[
\gg \frac{1}{F_{m-n} F_a} \gg \frac{1}{\gamma^{m-n} + a} \gg \frac{1}{\gamma^{2L}}. \tag{13}
\]
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Since \( n > x/(\log x)^2 \), it follows from estimates (12) and (13) that the lower bound

\[
\frac{\sigma(A)}{A} - 1 > \frac{1}{\gamma^4L}
\]

holds for large \( x \). Combining estimates (10) and (14), we get

\[
\frac{x}{(\log x)^2} \leq 2(\log \log x)^2 \gamma^4L = 2(\log \log x)^2 x^{2\log \gamma}.
\]

which is impossible for large \( x \) because \( 2 \log \gamma < 1 \). This completes the proof of the fact that \( A_6(x) \) is empty for large \( x \).

3. Further Remarks

In this note, we proved that for almost all positive integers \( n \), \( \sigma(F_n) \) is not a Fibonacci number, and by the result from [7] the same is true for \( \sigma(F_n) - F_n \). Recall that the Zeckendorf decomposition of the positive integer \( n \) is its representation

\[
n = F_{m_1} + \cdots + F_{m_t},
\]

where \( 0 < m_1 < \cdots < m_t \) and \( m_{i+1} - m_i \geq 2 \) for all \( i = 1, \ldots, t-1 \). It is known [9] that such a representation always exists and up to identifying \( F_2 \) with \( F_1 \), it is also unique. Let \( \ell(n) = t \) be the length of the Zeckendorf decomposition of \( n \). We conjecture that \( \ell(\sigma(F_n)) \) tends to infinity with \( n \) on a set of asymptotic density 1 and we would like to leave this question for the reader. Note that our main result shows that \( \ell(\sigma(F_n)) \geq 2 \) holds for almost all \( n \).

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