Aliquots sums of Fibonacci numbers

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Abstract

Here, we investigate the Fibonacci numbers whose sum of aliquot divisors is also a Fibonacci number (the prime Fibonacci numbers have this property).

1 Introduction

Let \((F_n)_{n \geq 1}\) be the sequence of Fibonacci numbers. For a positive integer \(n\) we write \(\sigma(n)\) for the sum of divisors function of \(n\). Recall that a number \(n\) is called \emph{multiply perfect} if \(n \mid \sigma(n)\). If \(\sigma(n) = 2n\), then \(n\) is called \emph{perfect}. In [2], it was shown that there are only finitely many multiply perfect Fibonacci

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numbers, and in [3], it was shown that no Fibonacci number is perfect. For a positive integer \( n \), the value \( \varphi(n) \) of the Euler function is defined to be the number of natural numbers less than or equal to \( n \) and coprime to \( n \).

Let \( s(n) = \sigma(n) - n \). The number \( s(n) \) is sometimes called the sum of aliquot divisors of \( n \). Two positive integers \( m \) and \( n \) (with \( m \neq n \)) are called amicable if \( s(m) = n \) and \( s(n) = m \). It is not known if there exist infinitely many amicable pairs, but Pomerance [5] showed that the sum of the reciprocals of all the members of all amicable pairs is convergent.

Here, we search for Fibonacci numbers \( F_n \) such that \( s(F_n) \) is a Fibonacci number. In particular, prime Fibonacci numbers have the above property. We put \( A = \{ n : s(F_n) = F_m \text{ for some positive integer } m \} \).

In this paper, we give an upper bound on the counting function of \( A \).

**Theorem 1.** There exists a positive constant \( c_0 \) such that the inequality

\[
\#A(x) < c_0 \frac{x}{\log \log \log x}
\]

holds for all \( x > e^{e^{e^e}} \).

Throughout this paper, we use the Vinogradov symbols \( \gg, \ll \) and the Landau symbols \( O, \asymp \) and \( o \) with their usual meanings. We recall that \( A \ll B, B \gg A \) and \( A = O(B) \) are all equivalent and mean that \( |A| < cB \) holds with some constant \( c \), while \( A \asymp B \) means that both \( A \ll B \) and \( B \ll A \) hold. For a positive real number \( x \) we write \( \log x \) for the maximum between 2 and the natural logarithm of \( x \). Note that with this convention, the function \( \log x \) is sub-multiplicative; i.e., the inequality \( \log(xy) \leq \log x \log y \) holds for all positive numbers \( x \) and \( y \). For a positive real number \( t \) and a subset \( B \) of the positive integers, we write \( B(t) = B \cap [1, t] \). We use \( p, q, P \) and \( Q \) with or without subscripts to denote prime numbers.

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# The Proof of Theorem 1

Let \( x \) be a large positive real number.
2.1 Some sieving

Let \( \omega(n) \) and \( \Omega(n) \) be the number of prime divisors of \( n \) and the number of prime power divisors of \( n \) (\( > 1 \)), respectively. Let

\[
\mathcal{A}_1(x) = \{ n \leq x : \omega(n) < 0.9 \log \log x \text{ or } \Omega(n) > 1.1 \log \log x \};
\]  
(1)

By the Turán-Kubilius inequalities (see [8])

\[
\sum_{n \leq x} (f(n) - \log \log x)^2 = O(x \log \log x)
\]

for both \( f \in \{ \omega, \Omega \} \),

we infer that

\[
\# \mathcal{A}_1(x) \ll \frac{x}{\log \log x}.
\]  
(2)

Let \( y = (\log \log x)^{1/3} \) and let

\[
\mathcal{A}_2(x) = \{ n \leq x : p \nmid n \text{ for all primes } p < y \}.
\]  
(3)

By Brun’s sieve,

\[
\# \mathcal{A}_2(x) \ll x \prod_{p < y} \left(1 - \frac{1}{p}\right) \ll \frac{x}{\log y} \ll \frac{x}{\log \log \log x}.
\]  
(4)

We now write

\[
\sigma(F_n) = F_n + F_m,
\]

and we look at bounds for \( m \) in terms of \( n \), where \( n \leq x \) does not belong to \( \mathcal{A}_1(x) \cup \mathcal{A}_2(x) \).

2.2 Bounds for \( m \) in terms of \( n \)

We start with a lower bound for \( m \). Let \( \gamma = (1 + \sqrt{5})/2 \) be the golden section. Let \( n \leq x \) not in \( \mathcal{A}_1(x) \cup \mathcal{A}_2(x) \). Then, there exists \( p < y \) such that \( p \nmid n \). Thus, \( F_p \nmid F_n \), therefore

\[
\gamma^m > F_m = s(F_n) \geq \frac{F_n}{F_p} \gg \gamma^{n-p} \geq \gamma^{n-y},
\]

where we used the fact that \( F_n \approx \gamma^n \). Hence,

\[
m \geq n - y + O(1),
\]

3
therefore 

\[ m \geq n - 2y, \]

once \( x \) is sufficiently large. We now look at an upper bound for \( m \). Note that

\[ \gamma^{m-n} \ll \frac{F_m}{F_n} \leq \frac{\sigma(F_n)}{F_n} \leq \prod_{p|F_n} \left( 1 + \frac{1}{p-1} \right). \quad (5) \]

For every prime number \( p \) let \( z(p) \) be its order of apparition in the Fibonacci sequence, and for a positive integer \( d \) let \( \mathcal{P}_d = \{ p : z(p) = d \} \). It is known that \( p \equiv \pm 1 \pmod{z(p)} \) holds for all primes \( p > 5 \) and it is clear that

\[ F_d \geq \prod_{p \in \mathcal{P}_d} p \gg (d-1)^{\# \mathcal{P}_d}, \]

therefore

\[ \# \mathcal{P}_d \ll \frac{d}{\log d}. \quad (6) \]

Furthermore, \( z(p) \gg \log p \). We now get by taking logarithms in (5) that

\[ m - n \leq \sum_{p|F_n} \frac{1}{p-1} + O(1) \leq \sum_{d|n} \sum_{p \in \mathcal{P}_d} \frac{1}{p-1} + O(1). \]

Obviously,

\[ \sum_{p \in \mathcal{P}_d} \frac{1}{p-1} \leq \sum_{p \equiv \pm 1 \pmod{d}} \frac{1}{p-1} + \frac{\# \mathcal{P}_d}{d^2 - 2} \ll \frac{\log \log d}{\varphi(d)}, \]

where in the above inequality we have used estimate (6) as well as the known fact that the inequality

\[ \sum_{p \equiv a \pmod{b}} \frac{1}{p-1} \leq \frac{1}{p_1(a,b) - 1} + O\left( \frac{\log \log t}{\varphi(b)} \right), \quad (7) \]

holds uniformly in coprime positive integers \( a < b \) and positive real numbers \( t \), where \( p_1(a,b) \) is the first prime in the arithmetic progression \( a \pmod{b} \).
(see, for example, [4]). Since the function \( \log \log d \) is sub-multiplicative, we get that

\[
m - n \leq \prod_{p^e \mid n} \left( 1 + O \left( \sum_{\nu=1}^{\mu} \frac{\log \log (p^\nu)}{p^\nu} \right) \right).
\]

\[
\leq \exp \left( O \left( \sum_{p \mid n} \frac{\log \log p}{p} + \sum_{p \geq 2} \sum_{\nu \geq 2} \frac{\log \log (p^\nu)}{p^\nu} \right) \right)
\]

\[
= \exp \left( O \left( \sum_{p \mid n} \frac{\log \log p}{p} + 1 \right) \right).
\]

Since \( n \not\in A_1(x) \), it follows that \( \omega(n) < 1.1 \log \log x \). Thus, if we write \( p_1 < p_2 < \ldots \) for the increasing sequence of all the prime numbers, then

\[
\sum_{p \mid n} \frac{\log \log p}{p} \leq \omega(n) \sum_{\nu=1}^{\mu} \frac{\log \log p}{p} \leq \int_2^{p_\omega(n)} \frac{\log \log t}{t} d\pi(t)
\]

\[
\ll (\log \log p_{\omega(n)})^2 \ll (\log \log \log \log x)^2.
\]

Hence,

\[
m - n \leq \exp \left( O((\log \log \log \log x)^2) \right) < 2y,
\]

where the last inequality holds if \( x \) is large. In conclusion, if \( n \leq x \) is not in \( A_1(x) \cup A_2(x) \), then \( m \in [n - 2y, n + 2y] \).

2.3 More sieving

Let \( Q = \{ q : z(q) < q^{1/3} \} \). Note that uniformly in \( t > 1 \),

\[
2^#Q(t) \leq \prod_{q \in Q \atop q < t} q \leq \prod_{n < t^{1/3}} F_n < \gamma \sum_{n < t^{1/3}} n < \gamma^{2/3},
\]

therefore

\[
#Q(t) \ll t^{2/3},
\]
which shows that
\[ \sum_{q \in \mathbb{Q}} \frac{1}{q} \leq \int_{s}^{\infty} \frac{1}{t} d\# \mathcal{Q}(t) \]
\[ \leq \frac{\# \mathcal{Q}(t)}{t} \bigg|_{t=s}^{t=\infty} + \int_{s}^{\infty} \frac{\# \mathcal{Q}(t)}{t^2} dt \]
\[ \leq \frac{1}{s^{1/3}} + \int_{s}^{\infty} \frac{dt}{t^{4/3}} \ll \frac{1}{s^{1/3}}. \] (8)

We now put \( u = (\log x)^3 \) and let
\[ \mathcal{A}_3(x) = \{ n \leq x : z(p)p \mid n \text{ for some } p > u \}. \] (9)

For every fixed prime \( p > u \), the number of \( n \leq x \) which are multiples of \( pz(p) \) is \( \lfloor x/pz(p) \rfloor \leq x/pz(p) \). So,
\[ \# \mathcal{A}_3(x) \leq \sum_{p > u} \frac{x}{pz(p)} \leq \sum_{p > u} \frac{x}{p} + \sum_{p > u} \frac{x}{z(p)p} \]
\[ \ll \sum_{u^{1/3} < d \leq x} \frac{x}{d^{1/3}} \sum_{p \equiv \pm 1 \pmod{d}} \frac{x}{dp} + \frac{x}{u^{1/3}} \]
\[ \ll x \sum_{u^{1/3} < d \leq x} \frac{\log \log d}{d\varphi(d)} + \frac{x}{u^{1/3}} \]
\[ \ll x \sum_{u^{1/3} < d \leq x} \frac{(\log \log d)^2}{d^2} + \frac{x}{u^{1/3}} \]
\[ \ll x(\log \log x)^2 \sum_{d > u^{1/3}} \frac{1}{d^2} + \frac{x}{u^{1/3}} \ll \frac{x(\log \log x)^2}{(\log x)^{1/3}}, \] (10)

where in the above estimates we used (8) with \( s = u^{1/3} \), the fact that \( \varphi(d) \gg d/\log \log d \) for all \( d \), as well as estimate (7) with \( b = d \) and \( a = 1 \) and \( d - 1 \), respectively.

We finally put \( \omega_u(n) \) for the number of prime factors \( p \leq u \) of \( n \), \( v = 2 \log \log \log x \) and let
\[ \mathcal{A}_4(x) = \{ n \leq x : \omega_u(n) > v \}. \] (11)
Again by Turán-Kubilius inequality,
\[ \sum_{n<x}(\omega_u(n) - \log \log u)^2 = O(x \log \log u), \]
and since \( \log \log u = (1 + o(1)) \log \log \log x \), we get easily that
\[ \#A_4(x) \ll \frac{x}{\log \log \log x}. \tag{12} \]

From now on, we deal only with numbers \( n \leq x \) which are not in \( A_1(x) \cup A_2(x) \cup A_3(x) \cup A_4(x) \).

### 2.4 The 2-adic order of \( \sigma(F_n) \)

Let \( K = \lfloor 0.8 \log \log x \rfloor \). Since \( n \notin \bigcup_{i=1}^4 A_i(x) \), we get that \( n \) has \( \omega(n) - \omega_u(n) > 0.9 \log \log x - 2 \log \log \log x > K \) prime factors \( P > u \), once \( x \) is sufficiently large. Let \( P_1 > P_2 > \ldots > P_K \) be the first (largest) prime factors of \( n \). Then \( P_K > u \). Note that
\[
F_n = \left( \prod_{i=0}^{K-1} \frac{F_{n/P_1\ldots P_i}}{F_{n/P_1\ldots P_{i+1}}} \right) F_{n/P_1\ldots P_K},
\]
where by convention we take \( P_0 = 1 \). Let
\[
L_i = \frac{F_{n/P_1\ldots P_i}}{F_{n/P_1\ldots P_{i+1}}} \quad \text{for} \quad i = 0, \ldots, K-1 \quad \text{and} \quad L_K = F_{n/P_1\ldots P_K}.
\]
We next observe that \( L_i \) and \( L_j \) are coprime for all \( 0 \leq i < j \leq K \). Indeed, assume that \( i < j \leq K \) and \( Q \) are such that \( Q \mid \gcd(L_i, L_j) \). Then
\[
Q \mid \gcd \left( \frac{F_{n/P_1\ldots P_{i+1}}}{F_{n/P_1\ldots P_{i+1}}}, \frac{F_{n/P_1\ldots P_i}}{F_{n/P_1\ldots P_{i+1}}} \right).
\]
However, it is well-known that the greatest common divisor appearing above divides \( P_{i+1} \). Hence, \( Q = P_{i+1} \), and \( Q \mid F_n \), therefore \( z(Q) \mid n \). Since \( Q > u > 5 \) for large \( x \), we get that \( Qz(Q) \mid n \) contradicting the fact that \( n \notin A_5(x) \). Thus, \( L_i \) and \( L_j \) are indeed coprime for all \( i < j \).

In [6], Ribenboim and McDaniel studied square-classes of Fibonacci numbers. Given two integers \( m \) and \( n \), they are in the same square-class if \( F_m F_n \)
is a square. It follows from their results that if \( m > 12n \) and \( n \) is sufficiently large, then \( m \) and \( n \) are not in the same square-class. In particular, if \( x \) is large, then none of the numbers \( L_i \) is a perfect square. Thus, there exists a prime \( Q_i \mid L_i \), such that the order at which \( Q_i \) appears in \( L_i \) (hence, in \( F_n \)) is odd. It is also clear that \( Q_i \) is odd if \( x \) is large enough (say if \( u > 3 \)). Thus, \( \prod_{i=1}^{K}(Q_i+1) \) is a divisor of \( \sigma(F_n) \), which proves that \( \sigma(F_n) \) is a multiple of \( 2K \).

2.5 The conclusion

Let \( A_5(x) \) be the set of all positive integers \( n \in A(x) \) which are not in \( \bigcup_{i=1}^{4} A_i(x) \). Let \( n_1 < n_2 < \ldots < n_{\ell} \) be all the elements in \( A_5(x) \). Then there exists \( k_i \in [-2y, 2y] \) such that \( m_i = n_i + k_i \) for all \( i = 1, \ldots, \ell \). Furthermore, \( 2K \mid \sigma(F_{n_i}) = F_{n_i} + F_{n_i+k_i} \). Let \( M = [4y + 1] \). We show that if \( \ell > M \), then \( n_i + M - n_i \) is large whenever \( i \leq \ell - M \). Indeed, let \( n_i < n_{i+1} < \ldots < n_{i+M} \). Then \( k_j \in [-2y, 2y] \) for all \( j = i, \ldots, i + M \), and since there are at most \( 2[2y] + 1 < M + 1 \) possible values of \( k_j \) and \( M + 1 \) possibilities for the index \( j \), it follows that there exist \( j_1 < j_2 \) in \( \{i, \ldots, i + M\} \) such that \( k_{j_1} = k_{j_2} \). Let \( k \) denote the common value of \( k_{j_1} \) and \( k_{j_2} \). Using the formula \( F_n = (\gamma^n - \delta^n)/(\gamma - \delta) \), where \( \delta = (1 - \sqrt{5})/2 \) is the conjugate of \( \gamma \), we note that the relation \( 2K \mid F_{n_{j_1}} + F_{n_{j_1}+k} \) gives

\[
\gamma^{n_{j_1}}(1 + \gamma^k) - \delta^{n_{j_1}}(1 + \delta^k) \equiv 0 \pmod{2^K},
\]

and similarly for \( n_{j_2} \). Here and in what follows, we say that an algebraic integer \( \alpha \) is a multiple of an integer \( m \) if \( \alpha/m \) is an algebraic integer. Write \( \lambda = n_{j_2} - n_{j_1} \). Then the above relation for \( n_{j_2} \) gives

\[
\gamma^{n_{j_1}}\gamma^{\lambda}(1 + \gamma^k) - \delta^{n_{j_1}}\lambda^{\lambda}(1 + \delta^k) \equiv 0 \pmod{2^K}.
\]

Multiplying the congruence (13) by \( \gamma^\lambda \) and subtracting it from congruence (14), we get that

\[
\delta^{n_{j_1}}(1 + \delta^k)(\gamma^\lambda - \delta^\lambda) \equiv 0 \pmod{2^K}.
\]

Conjugating the above relation (15) and multiplying the resulting congruences, we get

\[
|1 + \delta^k||1 + \gamma^k||\gamma^\lambda - \delta^\lambda|^2
\]

8
is an integer which is a multiple of $2^K$. Noting that the above integer is nonzero, by taking logarithms we get

$$\lambda \log \gamma + O(M) \geq 2K,$$

therefore $\lambda \gg K$. Thus, we just proved that $n_{i+M} - n_i \gg K$, therefore

$$\#A_5(x) \ll \frac{xM}{K} + M \ll \frac{x}{(\log \log x)^{2/3}}, \quad (16)$$

which together with the upper bounds (2), (4), (10) and (12) completes the proof of Theorem 1.

References


