2014-07

Signal-to-noise ratio limitations for intensity correlation imaging

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https://hdl.handle.net/10945/44137

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1. INTRODUCTION

The intensity correlation imaging (ICI) concept has been put forward as a possible approach to the task of obtaining, with ground-based equipment, a high-resolution image of satellites in a geosynchronous orbit. The ICI [1–3] concept is based on what we call the Hanbury Brown and Twiss (HBT) effect [4]. Because exploitation of the HBT effect relies on the use of intensity interferometry, which does not require maintaining a phase relationship between the optical signals collected by each of a pair of telescopes, this approach seems very appealing. That the HBT effect has not seen much use in astronomical circles is most likely due to the difficulty of achieving a suitable signal-to-noise ratio. This difficulty is made significantly worse for ICI by the fact that the target object, that is, the satellite being viewed, is significantly fainter than the stellar objects viewed by HBT. The ICI concept hopes to compensate for this difficulty by the greatness of the redundancy of the set of measurements collected—using multiple pairs of apertures and multiple spectral bands simultaneously, along with very many measurement instants. This paper addresses the question of just how great that redundancy would have to be to allow a suitable measurement result to be achieved in a reasonable amount of time. We intend to show that the required degree of redundancy would be prohibitively large.

As a first step in this process we briefly review the concept of intensity interferometry as established by the work of HBT.
An understanding of this concept starts with recognition of the fact that radiation coming from the target object is incoherent—both spatially and temporally incoherent. This incoherence is well modeled by considering the radiation coming from the target object to be due to a set of random oscillators—for each position on the target object there being a continuum of oscillators spanning the optical frequency domain. These oscillators are random in the sense that each oscillator’s output is defined by the randomly selected value of a complex phasor, with this phasor defining the amplitude and phase of the oscillator’s output. Calling the target object’s radiation incoherent corresponds to asserting that there is no correlation between any two of the random phasors except if the two oscillators have exceptionally small separation in location and in optical frequency. In the analysis that will be presented in this work this restriction of correlation to exceedingly proximate oscillators will be used to provide a basis for the carrying out of a combination of ensemble averaging and integration over position and frequency.

Because of the incoherence of the radiation from the target object the optical field collected by some telescope viewing the target object in some limited spectral band (i.e., over some limited range of optical frequencies) will be randomly varying. As a consequence the amount of optical energy that is collected by the telescope in some integration time will be a random quantity—a quantity whose value will vary randomly from one integration time to an other!

It should be recognized that this random variation is not normally detected or noticed when the strength of the radiation from an incoherent source, for instance a blackbody reference source, is measured. This is because of the very large value of the time–bandwidth product for the measurement. For example, if the measurement’s integration time were as short as $\Delta t = 1.0 \times 10^{-6}$ s and its bandwidth were as small as $\Delta \nu = 3.0 \times 10^{10}$ Hz (as it would be if the wavelength were $\lambda = 1.0 \times 10^{-6}$ m $\equiv$ 1.0 $\mu$m and the spectral bandwidth were $\Delta \lambda = 1.0 \times 10^{-10}$ m $\equiv$ 1.0 $\times$ 10$^{-4}$ $\mu$m $\equiv$ 1.0 $\AA$) then the time–bandwidth product would be $\Delta t \Delta \nu = 3.0 \times 10^{14}$. The measurement of the collected optical energy would, in effect, represent an average over 30,000 statistically independent random realizations of the randomly fluctuating optical power, so the random variation of the measurement result would be very small compared to the average value of the measurement result. Such a small fractional variation would be difficult to detect—especially because of shot-noise effects—but, small as it is and difficult as it is to detect, nonetheless it is present [5] and is potentially exploitable.

What is referred to here as the HBT effect is the fact that when two telescopes collect light in the same integration time and in the same spectral band from some target object of interest, there is a covariance between the random variations of the amount of optical energy collected by each telescope. Moreover, the value of this covariance is directly proportional to the value of the square of the amplitude of one of the components of the Fourier transform of the brightness pattern of the target object—the particular component of the Fourier transform being the one that goes with a spatial frequency whose value is set by the separation of the two telescopes.

The ICI concept calls for covariance values to be measured for a suitable set of separations between pairs of telescopes. From this set of covariance values a corresponding set of amplitudes of the components of the Fourier transform of the image of the target object are to be developed. Using a phase recovery algorithm [6,7] the corresponding set of phase values are to be recovered, and from this an image of the target object is to be developed by inverse Fourier transforming the data (or by some alternative method). In this work we are concerned only with the signal-to-noise ratio that is to be expected for any one of the measured covariance values.

There are at least four processes each of which, singly or collectively, will cause there to be an error in a measured covariance value. These four error producing processes are associated with (1) shot noise in the basic optical energy measurement; (2) the random nature of the process which the covariance of interest characterizes, which—because of the finite size of the set of sample values used in formulating an estimate of the value of that covariance—is only incompletely averaged (3) turbulence induced scintillation; and (4) the fact that the finite size of the telescope’s aperture and the spread of optical wavelengths that are used result in the measured covariance value actually corresponding to an average over a set of different covariance values. In this work we will consider only the shot-noise-related error—understanding that the value we will develop here for the signal-to-noise ratio that is to be associated with the estimated value of the covariance is only an upper limit. (Here and in what follows we use the term signal-to-noise ratio to denote the ratio of the expected value of the covariance that is being measured divided by the rms error that is to be associated with that measured value.)

2. STATISTICS OF AN INCOHERENT OPTICAL FIELD

Starting with the midpoint between a pair of telescopes and a point at the center of the target object, we take the line between those two points as defining the $z$ axis of our coordinate system. We shall refer to a plane perpendicular to this $z$ axis and proximate to the ground as the ground plane and will use the notation $\rho$; to denote positions on the ground plane. We shall assume that the two telescopes each have their apertures on that plane. We shall refer to the plane perpendicular to the $z$ axis and proximate to the target object as the target plane and will use the notation $r$ to denote positions on the target plane. We will use the notation $Z$ to denote the distance between the ground plane and the target plane.

We shall consider the pattern or image presented by the target object to exist on the target plane and will take this pattern to be the source of the optical field collected by the telescopes at the ground plane. We shall use the notation $J(r)$ to represent this pattern—the quantity $J(r)$ being a spectral power density and having the dimensions of W/m$^2$. Hz. It is to be noted that this quantity is nonrandom, in distinction to the optical field $\langle \cdots \rangle_{\text{inc}}$, which is random—the optical field being incoherent. To relate this optical source pattern, $J(r)$, to the incoherent optical field we write that

$$J(r) = \left\langle \frac{1}{2} |u(r, t)|^2 \right\rangle_{\text{inc}}. \quad (1)$$

where $u(r, t)$ denotes the random, incoherent optical field on the target plane at position $r$ and time $t$ and where the notation $\langle \cdots \rangle_{\text{inc}}$ indicates the process of forming an ensemble average
over the statistics of the incoherence of the optical field. Restricting attention to only a narrow spectral band, extending from a \(v_L\) to \(v_U\), we can express the incoherent optical field, \(u(r,t)\), in terms of a set of oscillators, writing that

\[
u(r,t) = \int_{v_L}^{v_U} d\nu a(\nu,r) \exp(2\pi i \nu t),
\]

(2)

with the notation \(a(\nu,r)\) being used to denote a randomly selected complex value—the phasor for the oscillator at position \(r\) with optical frequency \(\nu\). That the optical field is entirely incoherent corresponds to there being no correlation between the two phasors \(a(\nu,r)\) and \(a(\nu',r')\) unless the frequency difference, \(\nu - \nu'\), and the position difference, \(r - r'\), each have negligibly small magnitudes. From this statement it follows that

\[
\int_{v_L}^{v_U} d\nu \langle a(\nu,r) a^*(\nu',r)\rangle f(\nu',r) = f(\nu,r) \beta_\nu \langle |a(\nu,r)|^2 \rangle_{\text{inc}},
\]

(3)

and that

\[
\int d\nu' \int_{v_L}^{v_U} d\nu a(\nu,r) a^*(\nu',r') f(\nu',r') = f(\nu,r) \beta_\nu \beta_{\nu'} \langle |a(\nu,r)|^2 \rangle_{\text{inc}}.
\]

(4)

where \(f(\nu,r)\) is any reasonably well-behaved function of \(\nu\) and \(r\)—with the phrase reasonably well-behaved implying that the value of \(f(\nu,r)\) does not change significantly or noticeably when the value of \(\nu\) or of \(r\) changes by only a very small amount. The quantities \(\beta_\nu\) and \(\beta_{\nu'}\) appearing in Eq. (4) are measures of the range and of the level of correlation in \(\nu'\) and in \(r'\), respectively, over which there is a nonzero amount of correlation between the value of \(a(\nu',r')\) and the value of \(a(\nu,r)\). As will be shown below, the value of \(\beta_\nu \langle |a(\nu,r)|^2 \rangle_{\text{inc}}\) and of \(\beta_{\nu'} \langle |a(\nu,r)|^2 \rangle_{\text{inc}}\) can be calculated from consideration of the spectral power density on the target plane and on the ground plane.

It also follows from further consideration of the lead sentence of the above paragraph—or, in a more analytic way, from repeated application of Eq. (4)—that

\[
\Delta \nu \mathcal{F}(r) = \left(\frac{1}{2} |u(r,t)|^2\right)_{\text{inc}}.
\]

(6)

Expressing the absolute value squared in this equation as the product of \(u(r,t)\) times its complex conjugate, substituting Eq. (2) twice into this revised expression, once with the variable of integration shown as \(\nu\) and the second time with it shown as \(\nu'\), then making a double integral of the product of integrals, and after that interchanging the order of ensemble averaging and integration over \(\nu'\), we obtain a result which can be written as

\[
\Delta \nu \mathcal{F}(r) = \frac{1}{2} \int_{v_L}^{v_U} d\nu \left( \int_{v_L}^{v_U} d\nu' a(\nu,r) a^*(\nu',r) \exp(2\pi i (\nu - \nu') t) \right)_{\text{inc}}.
\]

(7)

Making use of Eq. (3) to carry out the ensemble averaging and the \(^{\nu'}\) integration—and noting that when \(\nu' = \nu\) then \(\exp(2\pi i (\nu - \nu') t) = 1\)—we obtain a result that can be written as

\[
\Delta \nu \mathcal{F}(r) = \frac{1}{2} \beta_\nu \int_{v_L}^{v_U} d\nu \langle |a(\nu,r)|^2 \rangle_{\text{inc}}.
\]

(8)
Arguing that with a very narrow spectral band one should expect the value of \(|\langle |\alpha(\nu, \mathbf{r})|^2 \rangle|_{\text{inc}}\) to be independent of the value of \(\nu\) and accordingly can carry out the \(\nu\)-integration (with an integrand having no \(\nu\) dependence), one obtains a result that can be written as

\[ 2\mathcal{F}(\mathbf{r}) = \beta_i \langle |\alpha(\nu, \mathbf{r})|^2 \rangle|_{\text{inc}}, \quad (9) \]

after canceling a factor of \(\Delta \nu\) on both sides of the equation and multiplying both sides by a factor of two.

For the evaluation of \(\beta_i\), we direct attention to the expected optical power density on the ground plane, which we denote by the notation \(\mathcal{P}\). Denoting the random optical field on the ground plane at position \(\rho\); and time \(t\) by the notation \(u(\rho, t)\), the expected optical power density is given by the equation

\[ \mathcal{P} = \left\{ \frac{1}{2} |v(\rho, t)|^2 \right\}_{\text{inc}}. \quad (10) \]

We now make use of the Fresnel–Kirchhoff formulation for the propagation of the optical field, \(u(\mathbf{r}, t)\), on the target plane to produce the optical field, \(v(\rho, t + \tau)\), on the ground plane—at the latter time \(t + \tau\), where \(\tau = Z/c\)—with the notation \(c\) denoting the speed of light so \(\tau\) is the time it takes light to travel from the target plane to the ground plane.

Letting \(S(\mathbf{r}, \rho)\) denote the distance between the position \(\rho\); on the target plane and the position \(\mathbf{r}\) on the ground plane, with \(Z\) being very much greater than \(|\mathbf{r} - \rho|\), we can write that

\[ S(\mathbf{r}, \rho) = \left[ Z + |\mathbf{r} - \rho|^2 \right]^{1/2} \approx Z + \frac{|\mathbf{r} - \rho|^2}{2Z} \approx Z + \frac{\rho^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{p}}{Z}. \]

(11)

With this approximation, applying the Fresnel–Kirchhoff formulation separately to each optical frequency component, \(\alpha(\nu, \mathbf{r})\) \(\exp(2\pi i \nu t)\), of the optical field, \(u(\mathbf{r}, t)\)—in this regard, cf. Eq. (2)—we can write that

\[ v(\rho, t + \tau) = \int_{\nu_i}^{\nu_f} \frac{\nu}{c} \int d\alpha(\nu, \mathbf{r}) \exp(2\pi i \nu t) \]

\[ \times \exp \left( 2\pi \frac{\nu}{c} \left[ Z + \frac{\mathbf{r}^2 + \rho^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{p}}{Z} \right] \right). \quad (12) \]

Now consider Eq. (10) with the optical field \(v(\mathbf{r}, t)\) replaced by the optical field \(v(\rho, t + \tau)\) (a change of no physical significance since it is within the ensemble averaging brackets), and with the square of the absolute value of the optical field replaced by the product of the optical field times its complex conjugate. Into this revised version of Eq. (10) we will substitute Eq. (12) twice—once with the variables of integration shown as \(\nu\) and \(\mathbf{r}\), and once with the variables of integration shown as \(\nu'\) and \(\mathbf{r}'\). We then will make a multiple integral of the product of integrals and will interchange the order of ensemble averaging and integration. Doing this we obtain a result which can be written as

\[ \mathcal{P} = \frac{1}{2} \int_{\nu_i}^{\nu_f} d\nu \int_{\nu_i}^{\nu_f} d\nu' \int d\alpha(\nu, \mathbf{r}) \alpha^*(\nu', \mathbf{r}') \]

\[ \times \exp(2\pi i (\nu - \nu') t) \]

\[ \times \exp \left( 2\pi \frac{\nu}{c} \left[ Z + \frac{\mathbf{r}^2 + \rho^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{p}}{Z} \right] \right) \]

\[ \times \exp \left( 2\pi \frac{\nu'}{c} \left[ Z + \frac{\mathbf{r}'^2 + \rho^2}{2Z} - \frac{\mathbf{r}' \cdot \mathbf{p}}{Z} \right] \right) \right\}_{\text{inc}}. \quad (13) \]

Making use of Eq. (4) we can carry out the ensemble averaging operation along with the \(\nu'\)- and \(\nu\)-integrations, obtaining the result that

\[ \mathcal{P} = \frac{1}{2} \int_{\nu_i}^{\nu_f} d\nu \int d\nu' \frac{\nu^2}{c^2 Z^2} \beta_i \beta_t \langle |\alpha(\nu, \mathbf{r})|^2 \rangle|_{\text{inc}}, \quad (14) \]

since both of the exponential functions in Eq. (13) take a value of unity when \(\nu' = \nu\) and \(\mathbf{r}' = \mathbf{r}\).

Recognizing that for the very narrow spectral band between \(\nu = \nu_i\) and \(\nu = \nu_f\), the value of \(\nu\) in the integrand of Eq. (14) can be replaced by \(\bar{\nu}\), and then substituting Eq. (9) into this formulation, on performing the \(\nu\)-integration we obtain the result that

\[ \mathcal{P} = \bar{\nu}^2 \Delta \nu \frac{\mathcal{P}}{\bar{\nu}} \int d\mathcal{F}(\mathbf{r}), \quad (15) \]

from which it follows that

\[ \beta_t = \frac{\bar{\nu}^2 Z^2}{\bar{\nu}^2} \frac{\mathcal{P}}{\mathcal{P}} \int d\mathcal{F}(\mathbf{r}). \quad (16) \]

Combining Eqs. (9) and (16) we can write that

\[ \beta_i \beta_t \langle |\alpha(\nu, \mathbf{r})|^2 \rangle|_{\text{inc}} = \frac{1}{\bar{\nu}^2} \frac{\mathcal{P}}{\bar{\nu}^2} \int d\mathcal{F}(\mathbf{r}) = \frac{1}{\bar{\nu}^2} \frac{\mathcal{P}}{\bar{\nu}^2} \frac{\mathcal{P}}{\bar{\nu}} \int d\mathcal{F}(\mathbf{r}), \quad (17) \]

where

\[ \mathcal{F}(\mathbf{r}) = \mathcal{F}(\mathbf{r}) / \int d\mathcal{F}(\mathbf{r}). \quad (18) \]

We shall refer to \(\mathcal{F}(\mathbf{r})\) as the normalized pattern of the target object—normalized in the sense that \(\int d\mathcal{F}(\mathbf{r}) = 1\)—so the zero-spatial-frequency-component of the Fourier transform of \(\mathcal{F}(\mathbf{r})\) is unity.

With knowledge of the normalized pattern of the target object, \(\mathcal{F}(\mathbf{r})\), on the target plane and of the power density on the ground plane, \(\mathcal{P}\), then—with Eq. (17) in hand—we can consider the quantity \(\beta_i \beta_t \langle |\alpha(\nu, \mathbf{r})|^2 \rangle|_{\text{inc}}\) to be directly calculable.

4. COVARIANCE OF THE RANDOMLY VARYING AMOUNTS OF ENERGY COLLECTED IN AN INTEGRATION TIME BY EACH OF A PAIR OF TELESCOPES

Consider a pair of telescopes each with aperture diameter \(D\), with aperture centers at positions \(\rho_1\) and \(\rho_2\), synchronously collecting optical power for an integration time of duration \(\Delta t\). We here undertake the determination of the covariance...
between the amounts of energy collected by each in an integration time. Using the notations \( \mathcal{E}_A \) and \( \mathcal{E}_B \) to denote these two amounts of energy, we define the covariance, \( C_{AB} \), by the expression

\[
C_{AB} = \langle \mathcal{E}_A \mathcal{E}_B \rangle_{\text{inc}} = \langle \mathcal{E}_A^2 \rangle_{\text{inc}} - \langle \mathcal{E}_A \rangle_{\text{inc}}^2.
\]

where

\[
\langle \mathcal{E}_A \rangle_{\text{inc}} = \langle \mathcal{E}_B \rangle_{\text{inc}}.
\]

(19)

For evaluation of the covariance, \( C_{AB} \), we shall need to evaluate the quantity \( \langle \mathcal{E}_A \mathcal{E}_B \rangle_{\text{inc}} \)—but first we shall evaluate the quantity \( \tilde{\mathcal{E}} \), the mean measurement value.

Starting from Eq. (12) and proceeding from there in the same manner that led to Eq. (13), only in this case not considering the ensemble averaging process but rather taking account of the need to integrate over time (i.e., taking account of the detector system’s integration time, \( \Delta t \)) and also integrating over all positions in the telescope’s aperture, we can write that

\[
\mathcal{E}_A = \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \int_{-\Delta t/2}^{+\Delta t/2} dt \int d\nu d\nu' \int dr \int \frac{dr'}{c^2 Z^2} \alpha(\nu, r) \alpha^*(\nu', r') \times \exp \left( 2\pi i(\nu - \nu') t \right)
\]

\[
\times \exp \left( 2\pi i \left[ \frac{\nu}{c} \left( Z + \frac{r^2 + \rho_A^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{r}_A}{Z} \right) \right] \left[ \frac{\nu'}{c} \left( Z + \frac{r'^2 + \rho_B^2}{2Z} - \frac{\mathbf{r}' \cdot \mathbf{r}_B}{Z} \right) \right] \right).
\]

(20a)

\[
\mathcal{E}_B = \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \int_{-\Delta t/2}^{+\Delta t/2} dt \int d\nu d\nu' \int dr \int \frac{dr'}{c^2 Z^2} \alpha(\nu, r) \alpha^*(\nu', r')
\]

\[
\times \exp \left( 2\pi i(\nu - \nu') t \right)
\]

\[
\times \exp \left( 2\pi i \left[ \frac{\nu}{c} \left( Z + \frac{r^2 + \rho_A^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{r}_A}{Z} \right) \right] \left[ \frac{\nu'}{c} \left( Z + \frac{r'^2 + \rho_B^2}{2Z} - \frac{\mathbf{r}' \cdot \mathbf{r}_B}{Z} \right) \right] \right).
\]

(20b)

These formulations for \( \mathcal{E}_A \) and \( \mathcal{E}_B \) really should have, in place of the factor of \( (1/4)\pi D^2 \), an explicitly shown integration over the area of the telescopes aperture—having the quantities \( \rho_A \) and \( \rho_B \) (and their magnitudes squared) replaced by a variable that varies with position over the area of the aperture. However, based on the presumption that the aperture diameter, \( D \), is much too small to allow the telescope to even resolve the overall size of the target object, it can be shown that the value of the integrand is not significantly affected by simply using the quantities \( \rho_A \) and \( \rho_B \).

Taking the ensemble average over the statistics of the optical field’s incoherence on either form of Eq. (20) to obtain an expression for \( \tilde{\mathcal{E}} \), we here again make use of Eq. (4) to allow the \( \nu' \) and \( \mathbf{r}' \)-integrations to be performed. When these integrations are performed the \( \nu' \) and \( \mathbf{r}' \) dependences in the integrand are replaced by \( \nu \) and \( \mathbf{r} \) dependences—as a consequence of which the two exponential functions in the integrand take values of unity. Then, since there is no \( t \)-dependence in the integrand, the \( t \)-integration can be performed—yielding a factor of \( \Delta t \). The result can then be written as

\[
\tilde{\mathcal{E}} = \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \Delta t \int_{-\Delta t/2}^{+\Delta t/2} dt \int dr \int \frac{dr'}{c^2 Z^2} \beta \beta' \langle (\alpha(\nu, r))^2 \rangle_{\text{inc}}.
\]

(21)

As before, taking account of the fact that the spectral band is very narrow allows the \( \nu \)-integration to be performed—replacing the \( \nu \) dependence by a dependence on \( \tilde{\nu} \) and introducing a factor of \( \Delta \nu \). Carrying this out and making use of Eq. (17) we can write that

\[
\tilde{\mathcal{E}} = \left( \frac{1}{4} \pi D^2 \right) \Delta t \frac{c^2 Z^2}{\tilde{\nu}^2} \tilde{\rho}.
\]

(22)

since the integration over \( r \) has as its integrand only the term \( J(\nu) \) [from the numerator in Eq. (17)] and so is canceled by the \( r' \)-integration in the denominator of Eq. (17).

Before starting an evaluation of the quantity \( \langle \mathcal{E}_A \mathcal{E}_B \rangle_{\text{inc}} \) it is convenient to note that

\[
\int_{-\Delta t/2}^{+\Delta t/2} dt \exp(2\pi i(\nu - \nu') t) = \Delta t \sin(\pi(\nu - \nu') \Delta t),
\]

(23)

where \( \sin(\nu) = \sin(x)/x \). Using this we can carry out the \( t \)-integration in Eq. (20), obtaining the results that

\[
\mathcal{E}_A = \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \Delta t \int_{-\Delta t/2}^{+\Delta t/2} dt \int d\nu d\nu' \int dr \int \frac{dr'}{c^2 Z^2} \alpha(\nu, r) \alpha^*(\nu', r')
\]

\[
\times \sin(\pi(\nu - \nu') \Delta t)
\]

\[
\times \exp \left( 2\pi i \left[ \frac{\nu}{c} \left( Z + \frac{r^2 + \rho_A^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{r}_A}{Z} \right) \right] \left[ \frac{\nu'}{c} \left( Z + \frac{r'^2 + \rho_B^2}{2Z} - \frac{\mathbf{r}' \cdot \mathbf{r}_B}{Z} \right) \right] \right).
\]

(24a)

\[
\mathcal{E}_B = \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \Delta t \int_{-\Delta t/2}^{+\Delta t/2} dt \int d\nu d\nu' \int dr \int \frac{dr'}{c^2 Z^2} \alpha(\nu, r) \alpha^*(\nu', r')
\]

\[
\times \sin(\pi(\nu - \nu') \Delta t)
\]

\[
\times \exp \left( 2\pi i \left[ \frac{\nu}{c} \left( Z + \frac{r^2 + \rho_A^2}{2Z} - \frac{\mathbf{r} \cdot \mathbf{r}_A}{Z} \right) \right] \left[ \frac{\nu'}{c} \left( Z + \frac{r'^2 + \rho_B^2}{2Z} - \frac{\mathbf{r}' \cdot \mathbf{r}_B}{Z} \right) \right] \right).
\]

(24b)

For the evaluation of the quantity \( \langle \mathcal{E}_A \mathcal{E}_B \rangle_{\text{inc}} \) we start by forming the product of the right-hand sides of Eqs. (24a) and (24b) within the incoherence ensemble averaging brackets, but replacing the notations for two of the variables of integration for Eq. (24a) shown there as \( \nu' \) and \( \mathbf{r}' \) by variables with notation \( \nu'' \) and \( \mathbf{r}'' \) respectively—and also replacing the notations for all four of the variables of integration for Eq. (24b) shown there as \( \nu, \nu', \mathbf{r}, \) and \( \mathbf{r}' \) by variables with notation \( \nu'', \nu''', \mathbf{r}'', \) and \( \mathbf{r}''' \), respectively. Treating the product of integrals as a multiple integral on the product of the integrand and then interchanging the order of ensemble averaging and certain of the integrations, we can write that...
\[ \langle \mathbb{E}_A \mathbb{E}_B \rangle_{\text{inc}} = \frac{1}{4} \left( \frac{1}{4} \pi D^2 \right)^2 (\Delta t)^2 \int_{v_i}^{v_f} dv' \int_{r_i}^{r_f} dr \frac{v'}{c^2 Z^2} \int_{v_i}^{v_f} dv'' \int_{r_i}^{r_f} dr'' \frac{v''}{c^2 Z^2} \]
\times \alpha(v, r) \alpha(v', r') \alpha'(v, r') \alpha'(v', r') \text{sinc} (\pi (v - v') \Delta t) \text{sinc} (\pi (v' - v'') \Delta t) \]
\times \exp \left( 2 \pi i \left[ \frac{\nu}{c} \left( \frac{\rho_A^2 - \rho_B^2}{2Z} - \frac{r \cdot (\rho_A - \rho_B)}{Z} \right) - \frac{\nu'}{c} \left( \frac{\rho_A^2 - \rho_B^2}{2Z} - \frac{r' \cdot (\rho_A - \rho_B)}{Z} \right) \right] \right) \].

(25)

Making use of Eq. (5) the ensemble averaging along with the \( v'', v''', r'', \) and \( r''' \) integrations can all be carried out. This yields a result which can be expressed as the sum of two terms—which terms we shall denote by the notations \( Q_1 \) and \( Q_2 \), writing

\[ \langle \mathbb{E}_A \mathbb{E}_B \rangle_{\text{inc}} = Q_1 + Q_2. \]

(26)

We shall take the \( Q_1 \) term to correspond to the part of the right-hand side of Eq. (5) for which the nonvanishing of the ensemble average is based on the portion of the range of integration on the variables \( v', v''', r', \) and \( r''' \), for which \( (v', r') \) is in very close proximity to \( (v, r) \), and \( (v''', r''') \) is in very close proximity to \( (v', r') \)—with the \( Q_2 \) term corresponding to the other part of the right-hand side of Eq. (5), the part for which the nonvanishing of the ensemble average is based on the portion of the range of integration on \( v', v''', r', \) and \( r''' \) for which \( (v', r') \) is in very close proximity to \( (v, r) \), and \( (v''', r''') \) is in close proximity to \( (v', r') \).

For the \( Q_1 \) term it can be seen that with \( (v', r') \approx (v, r) \) and \( (v''', r''') \approx (v', r') \) the exponential functions and the sinc functions each have essentially unity value so we can write that

\[ Q_1 = \frac{1}{4} \left( \frac{1}{4} \pi D^2 \right)^2 (\Delta t)^2 \int_{v_i}^{v_f} dv' \int_{r_i}^{r_f} dr \frac{v'^2}{c^2 Z^2} \beta_i \beta'_i \]
\times \langle |(v, r)|^2 \rangle_{\text{inc}} \langle |(v', r')|^2 \rangle_{\text{inc}}. \]

(27)

This can be recast as

\[ Q_1 = \left[ \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \Delta t \int_{v_i}^{v_f} dv \int_{r_i}^{r_f} dr \frac{v^2}{c^2 Z^2} \beta_i \beta_i \langle |(v, r)|^2 \rangle_{\text{inc}} \right] \]
\times \left[ \frac{1}{2} \left( \frac{1}{4} \pi D^2 \right) \Delta t \int_{v_i}^{v_f} dv' \int_{r_i}^{r_f} dr' \frac{v'^2}{c^2 Z^2} \beta_i \beta'_i \langle |(v', r')|^2 \rangle_{\text{inc}} \right].

(28)

Comparing each of the square-bracket terms in Eq. (28) with the right-hand side of Eq. (21) it can be seen that

\[ Q_1 = \tilde{\mathbb{g}}^2. \]

(29)

Considering this in conjunction with Eqs. (19) and (26) it can be seen that

\[ C_{A,B} = Q_2. \]

(30)

For the \( Q_2 \) term it can be seen that with \( (v', r') \approx (v, r) \) and \( (v''', r''') \approx (v', r') \) all of the exponential function of the integrand of Eq. (25) cancels except for the part indicating a dependence on \( \rho_A \) or on \( \rho_B \), while the two sinc functions take the same form. Accordingly, we can write that

\[ Q_2 = \frac{1}{4} \left( \frac{1}{4} \pi D^2 \right)^2 (\Delta t)^2 \int_{v_i}^{v_f} dv' \int_{r_i}^{r_f} dr \frac{v'^2}{c^2 Z^2} \beta_i \beta'_i \langle |(v, r)|^2 \rangle_{\text{inc}} \]
\times \langle |(v', r')|^2 \rangle_{\text{inc}} \text{sinc}^2 (\pi (v - v') \Delta t) \]
\times \exp \left( 2 \pi i \left[ \frac{v}{c} \left( \frac{\rho_A^2 - \rho_B^2}{2Z} - \frac{r \cdot (\rho_A - \rho_B)}{Z} \right) \right] \right) \].

(31)

Making some simplifications in the form of the argument of the exponential function in the integrand shown in Eq. (21) and taking note of Eq. (30) we can write that

\[ C_{A,B} = \frac{1}{4} \left( \frac{1}{4} \pi D^2 \right)^2 (\Delta t)^2 \int_{v_i}^{v_f} dv' \int_{r_i}^{r_f} dr \frac{v'^2}{c^2 Z^2} \beta_i \beta'_i \langle |(v, r)|^2 \rangle_{\text{inc}} \]
\times \langle |(v', r')|^2 \rangle_{\text{inc}} \text{sinc}^2 (\pi (v - v') \Delta t) \]
\times \exp \left( 2 \pi i \left[ \frac{v - v'}{c} \left( \frac{\rho_A^2 - \rho_B^2}{2Z} - \frac{(r - r') \cdot (\rho_A - \rho_B)}{cZ} \right) \right] \right). \]

(32)

Changing the variables of integration from \( r, r', v, \) and \( v' \) to the sum and difference quantities \( r_+, r_-, \nu_+, \) and \( \nu_- \)—where

\[ r_+ = \frac{1}{2} (r + r'), \quad r_- = r - r', \]
\[ \nu_+ = \frac{1}{2} (v + v'), \quad \text{and} \quad \nu_- = v - v', \]

and noting that \( v r - v' r' = \nu_r r_+ + \nu_- r_- \) and that \( v^2 v'^2 = (\nu_+^2 - \nu_-^2)^2 \)—we can recast Eq. (32) as
with the upper and lower limits of the \( \nu \)-integration having values of \( \mathcal{Z}_u = \nu_U - \nu_L - |\nu_U + \nu_L| \) and \( \mathcal{Z}_l = -\mathcal{Z}_u = |\nu_U + \nu_L - 2\nu_L + \nu_U + \nu_L| \). Considering the \( \nu \)-dependence of the integrand in Eq. (34) is completely dominated by the sinc\(^2\) function in that integrand and that with respect to the rest of the integrand's \( \nu \)-dependence this sinc\(^2\) function is rather like a Dirac delta function centered at \( \nu = 0 \). Noting except for a very narrow range of values of \( \nu_L \)—a range of values of about a few times \( 1/\Delta t \) above \( \nu_L \) and a few times \( 1/\Delta \nu \) below \( \nu_U \)—the limits on the \( \nu \)-integration might as well be \(-\infty\) and \(+\infty\) since the sinc\(^2\) function will have a negligibly small value outside that range, and further noting that

\[
\int_{-\infty}^{+\infty} d\xi \, \text{sinc}^2(\pi \xi K) = K^{-1},
\]

it can be seen that from Eq. (34) the result can be obtained that

\[
C_{AB} = \frac{1}{4} \left( \frac{1}{4} \pi D^2 \right)^2 \Delta t \int_{\nu_L}^{\nu_U} d\nu \int d\nu_+ \int d\nu_- \frac{\nu_+^2 \nu_-^2}{c^2 Z^4} \beta_1^2 \beta_2^2 \times \left( \left| a \left( \nu, \nu_+, \nu_- + \frac{1}{2} \nu \right) \right|^2 \right)_{\text{inc}} \\
\times \left( \left| a \left( \nu, \nu_+ + \frac{1}{2} \nu \right) \right|^2 \right)_{\text{inc}} \times \exp \left( -2\pi i \frac{\nu_+ \nu_- \cdot (\rho_A - \rho_B)}{cZ} \right).
\]

In writing this we have replaced the \( \rho_\nu \) notation by \( \rho \)—as well as having set all of the \( \nu \)-dependence to zero based on the fact that the sinc\(^2\) function appearing in Eq. (34) is being treated as a Dirac delta function in \( \nu \)-space, one centered on \( \nu = 0 \).

Since the value of \( \Delta \nu \) is so small that the values of \( \left| a(\nu, \nu_+, (1/2) \nu) \right|^2 \) and of \( \left| a(\nu, \nu_-, (1/2) \nu) \right|^2 \) can be considered to be independent of the precise value of \( \nu \) and since for values of \( \Delta \nu, \nu_+, \) and \( \rho_A - \rho_B \) that might be of interest the quantity \( \Delta \nu \cdot (\rho_A - \rho_B)/(cZ) \) has a value very much less than unity, we can replace the \( \nu \)-dependence in Eq. (36) by a dependence on \( \tilde{\nu} \)—and then can carry out the \( \nu \)-integration (obtaining a factor of \( \Delta \nu \)) and accordingly can write that

\[
C_{AB} = \frac{1}{4} \left( \frac{1}{4} \pi D^2 \right)^2 \Delta \nu \int d\nu \int d\nu_+ \int d\nu_- \frac{\nu_+^2 \nu_-^2}{c^2 Z^4} \beta_1^2 \beta_2^2 \times \left( \left| a \left( \tilde{\nu}, \nu_+, \nu_- + \frac{1}{2} \nu \right) \right|^2 \right)_{\text{inc}} \\
\times \left( \left| a \left( \tilde{\nu}, \nu_+ + \frac{1}{2} \nu \right) \right|^2 \right)_{\text{inc}} \times \exp \left( -2\pi i \frac{\nu_+ \nu_- \cdot (\rho_A - \rho_B)}{cZ} \right).
\]

Changing the variables of integration back from \( \nu_+ \) and \( \nu_- \) to \( \nu \) and \( \nu' \) [cf. Eq. (33)], then separating the double integral into a product of integrals, and finally taking note Eq. (17), we can develop from Eq. (37) the result that

\[
C_{AB} = \left( \frac{1}{4} \pi D^2 \right)^2 \frac{\Delta \nu}{\Delta t} \left( \int d\nu \exp \left( -2\pi i \frac{\nu \cdot (\rho_A - \rho_B)}{cZ} \right) \tilde{\mathcal{F}}(\nu) \right) \\
\times \left( \int d\nu' \exp \left( +2\pi i \frac{\nu' \cdot (\rho_A - \rho_B)}{cZ} \right) \tilde{\mathcal{F}}'(\nu') \right).
\]

The two curly bracket terms in Eq. (38) can be seen to be the Fourier transform and the complex conjugate of the Fourier transform of the normalized pattern of the target object, \( \tilde{\mathcal{F}}(\nu) \), for spatial frequency \( \kappa = \nu \rho_A/(\nu Z) \). Denoting the Fourier transform of the normalized pattern of the target object, \( \tilde{\mathcal{F}}(\nu) \) (or the notation \( \tilde{\mathcal{F}}(\nu) \)—a quantity which we define by the equation

\[
\tilde{\mathcal{F}}(\nu) = \int d\nu \exp(-2\pi i \nu \cdot r) \tilde{\mathcal{F}}(\nu),
\]

we can recast Eq. (38) as

\[
C_{AB} \left( \frac{1}{4} \pi D^2 \frac{\Delta \nu}{\Delta t} \right)^2 \left| \tilde{\mathcal{F}} \left( \rho_A - \rho_B \right) \lambda Z \right|^2,
\]

where

\[
\lambda = \frac{c}{\nu}.
\]

We call attention to the fact that the quantity \( (1/4) \pi D^2 \frac{\Delta \nu}{\Delta t} \) appearing in Eq. (40) is the amount of optical energy expected to be collected by a single telescope in a single integration time.

As given by Eq. (40) the covariance, \( C_{AB} \), is a covariance between two different randomly varying amounts of collected energy, so \( C_{AB} \) has units of joules squared. To facilitate comparison with shot-noise effects it is convenient to consider this covariance as being between two different randomly varying numbers of collected photons—or if one assumes a unity value quantum efficiency for the detectors—between two different randomly varying numbers of pds. Such a transition to a discussion in terms of pde rather than joules, with an assumption of unity quantum efficiency, is reasonable since typical deviations from unity quantum efficiency will have only a slight effect on the conclusions reached in this analysis—a slight effect which if properly taken account of would make the conclusion reached a bit stronger (i.e., the calculated value of the signal-to-noise ratio would be somewhat smaller).

Expressed in terms of pde the covariance is

\[
C_{AB} = \frac{\left( \frac{1}{4} \pi D^2 \Delta \nu \right)^2}{\Delta t} \left| \tilde{\mathcal{F}} \left( \rho_A - \rho_B \right) \lambda Z \right|^2,
\]

where \( h = 6.63 \times 10^{-34} \text{ J} \cdot \text{s} \) is Planck’s constant.

5. ANALYSIS OF SHOT-NOISE

As sample numerical calculations will shortly make clear, for an IC or type application of the HBT effect the expected amount of optical energy collected by a telescope in a single
integration time correspond to less than that of a single photon, so the expected number of pde for each measurement is less than one. Accordingly, a first question to be addressed is whether the value of the covariance between the amounts of energy collected by each of two telescopes during the same integration time, that is, the quantity denoted by \( C_{AB} \) appearing in Eq. (41), can be extracted from such signal-deficient data. (As will be seen, it can be.) The second and more substantial question concerns the shot noise that goes with the pde process and the consequent rms error in the measured value of the covariance.

We will use the notation \( \langle \ldots \rangle_{\text{pde}} \) to denote the forming of an ensemble average over the Poisson statistics that governs the pde process—for some particular expected number of pde, understanding that the expected number of pdes may be a quantity which (though having a specific value for a specific integration time) is randomly varying from one integration time to another as a result of the incoherence of the optical source. The overall random variation of the number of pde during an integration time is a doubly stochastic process \([8,9]\). We will use the notations \( x_n^A \) and \( x_n^B \) to denote the random number of pde to the A telescope and with the B telescope during the \( n^{\text{th}} \) integration time and will use the notations \( \xi_n^A \) and \( \xi_n^B \) to denote the corresponding expected numbers of pde—recognizing that \( \xi_n^A \) and \( \xi_n^B \) random variables with probability distributions governed by the incoherent nature of the optical source.

It is well known that, in accordance with their being governed by Poisson statistics,

\[
\langle x_n^A \rangle_{\text{pde}} = A \xi_n^A, \quad \langle x_n^B \rangle_{\text{pde}} = B \xi_n^B,
\]

\[
\langle x_n^A x_n^B \rangle_{\text{pde}} = A \xi_n^A + B \xi_n^B.
\] (43)

Averaging over the statistics of the incoherence of the optical field we can write that

\[
C_{AB} = \langle A \xi_n^A - \mu_B \xi_n^B \rangle_{\text{inc}} = \langle A \xi_n^A B \xi_n^B \rangle_{\text{inc}} - \mu^2,
\]

where \( \langle \xi_n^A \rangle_{\text{inc}} = \langle \xi_n^B \rangle_{\text{inc}} = \mu, \) (44)

with

\[
\mu = \frac{n D^2 \bar{\beta} \Delta t}{h}. \] (45)

—the quantity \( \mu \) denoting the expected number of pde per integration time.

Given some set of \( n = \{1, 2, 3, \ldots, N\} \) separate integration times and pairs of values for \( x_n^A \) and \( x_n^B \) we use the formula

\[
\hat{C}_{AB} = \frac{1}{N-1} \left( \sum_{n=1}^{N} A x_n^B x_n^A \right) - \frac{1}{N(N-1)} \left( \sum_{n=1}^{N} A x_n^2 \right) \left( \sum_{n=1}^{N} B x_n^2 \right). \] (46)

to form the estimate, \( \hat{C}_{AB} \), of the value of the covariance, \( C_{AB} \).

Enclosing this quantity in double angle brackets, that is, writing \( \langle \hat{C}_{AB} \rangle_{\text{pde}} \)—applying the double angle brackets on the right-hand side of Eq. (46) as well as on the left-hand side, then writing the angle-bracket average of a difference of two terms as a difference of the angle-bracket average of the two terms, making a double sum of the product of two sums, and finally expressing the double sum as the sum of two summations—
one over all the summand terms for which the two indices take the same value and the other over all the summand terms for which the two indices take different values—we can write that

\[
\langle \hat{C}_{AB} \rangle_{\text{pde}} = \frac{1}{N-1} \left( \sum_{n=1}^{N} A x_n^B x_n^A \right)_{\text{pde}} - \frac{1}{N(N-1)} \left( \sum_{n,n'=1}^{N} A x_n^B x_{n'}^A \right)_{\text{pde}} \]

\[
- \frac{1}{N(N-1)} \left( \sum_{n=1}^{N} A x_n^B x_n^A \right)_{\text{pde}}. \] (47)

Combining the first and third terms of Eq. (47), interchanging the order of summation and ensemble averaging, recognizing that the pde statistics of \( x_n^A \) and \( x_n^B \) are entirely independent, and making use of Eq. (42) to allow the ensemble average over pde statistics to be carried out, we can write that

\[
\langle \hat{C}_{AB} \rangle_{\text{pde}} = \frac{1}{N} \sum_{n=1}^{N} \langle A \xi_n^A B \xi_n^B \rangle_{\text{inc}} \]

\[
- \frac{1}{N(N-1)} \sum_{n,n'=1}^{N} \langle A \xi_n^A B \xi_n^B \rangle_{\text{inc}}. \] (48)

Since, with regard to the statistics of the incoherent optical field, there is no correlation between the random values of \( \xi_n^A \) and \( \xi_n^B \) when \( n \neq n' \)—so that \( \langle A \xi_n^A B \xi_n^B \rangle_{\text{inc}} = \langle A \xi_n^A \rangle_{\text{inc}} \langle B \xi_n^B \rangle_{\text{inc}} = \mu^2 \) for \( n \neq n' \)—and since there are exactly \( N \) terms in the first summation and exactly \( N(N-1) \) terms in the second summation and no dependence on the value of \( n \) (or of \( n' \)) in either summation, we can obtain from Eq. (48) the result that

\[
\langle \hat{C}_{AB} \rangle_{\text{pde}} = \langle A \xi_n^A \rangle_{\text{inc}} \langle B \xi_n^B \rangle_{\text{inc}} - \mu^2. \] (49)

From consideration of this in conjunction with Eq. (44) it thus can be seen that the expression for \( \hat{C}_{AB} \) given by Eq. (46) represents an unbiased estimator for the value of the covariance, \( C_{AB} \). This result resolves the first shot-noise-related question—whether the covariance between the amounts of energy collected by each of two telescopes during the same integration time can be extracted from such signal-deficient data—resolves the question positively; it can, no matter how signal-deficient the data.

This brings us to the second shot-noise-related question, in a sense the key question—what is the rms error in the measured value of the covariance. With the objective of simplifying this error analysis as much as possible we take advantage of the fact that our interest is in developing a lower limit value for the rms error and consider the case for which there actually is no incoherent optical field induced signal strength variations and consider only the shot-noise contribution to the estimate of the covariance if Eq. (46) were used to form
an estimate of the covariance—which in this case would be equal to zero. For this case the variance in the estimated value of the covariance would be

$$\text{Var}\{\hat{C}_{AB}\} = \left[ \frac{1}{N(N-1)} \left( \sum_{n=1}^{N} \left( \sum_{m=1}^{N} A_{nm}^{2} \right) \right) \right] \times \left[ \frac{1}{N(N-1)} \left( \sum_{n=1}^{N} \left( \sum_{m=1}^{N} B_{nm}^{2} \right) \right) \right].$$

Multiplying the square bracket terms and expressing a product of summations as a multiple summation, there are four terms—one being a double summation, two being triple summations, and one being a quadruple summation. With a simple interchange of the notations for the summation indices one of the triple summations can be made to have a form identical to the other triple summation. Then expressing the pde-ensemble average of the sum or difference of three terms as the corresponding sum or difference of pde ensemble averages of the three terms, we obtain from Eq. (50) the result that

$$\text{Var}\{\hat{C}_{AB}\} = \frac{S_2}{(N-1)^2} - \frac{2S_3}{N(N-1)^2} + \frac{S_4}{N^2(N-1)^2},$$

where

$$S_2 = \left( \sum_{n,m=1}^{N} A_{nm}^{4} \right)_{\text{pde}}.$$  \hspace{1cm} (52a)

$$S_3 = \left( \sum_{n,m,n',m'=1}^{N} A_{nm}^{2} A_{n'm'}^{2} B_{nm}^{2} B_{n'm'}^{2} \right)_{\text{pde}}.$$  \hspace{1cm} (52b)

$$S_4 = \left( \sum_{n,m,n',m'=1}^{N} A_{nm}^{2} A_{n'm'}^{2} B_{nm}^{2} B_{n'm'}^{2} \right)_{\text{pde}}.$$  \hspace{1cm} (52c)

For the reduction of $S_2$, starting from Eq. (52a), we first separate the double sum (of $N^2$ terms) into a single sum (on the $N$) of those terms for which both indices take the same value, plus the double sum (on the $N(N - 1)$) of those terms for which the two indices take different values—and then interchange the order of ensemble averaging and summation, allowing use to be made of the Poisson statistics expressed by Eq. (45), after which use is made of the fact that since there is no incoherent random variation being considered in this calculation so that $A_{nm}^{2} = B_{nm}^{2} = \mu$ and $A_{nm}^{2} = B_{nm}^{2} = \mu^2$. Accordingly, we write that

$$S_2 = \sum_{n=1}^{N} \left[ (A_{nm}^{2})_{\text{pde}} (B_{nm}^{2})_{\text{pde}} \right] + \sum_{n=1}^{N} \left[ (A_{nm}^{2})_{\text{pde}} (B_{n'm'}^{2})_{\text{pde}} \right].$$

For the evaluation of $S_3$, starting from Eq. (52b) and proceeding in essentially the same way as in developing Eq. (53) only this time having to separate the triple sum into four summations—the first a single summation in which $n = n' = n''$, the second a double summation in which $n' = n$ and $n'' \neq n$, the third a double summation in which $n' = n''$ and $n'' \neq n$, and the fourth a triple summation in which $n' \neq n$ and $n'' \neq n$—we obtain the result that

$$S_3 = N^3 \mu^4 + 2N^2 \mu^3 + N \mu^2.$$  \hspace{1cm} (54)

For the evaluation of $S_4$ it is convenient to first modify the quadruple sum of Eq. (52c) into a product of two double sums, the first over $n$ and $n''$ and the second over $n'$ and $n'''$ and then, since the pde statistics of the summands of these two double sums are statistically independent, express the pde-ensemble average of the product of the two sums as the product of the pde-ensemble averages of each of these two sums, writing

$$S_4 = \left( \sum_{n,m=1}^{N} A_{nm}^{4} \right)_{\text{pde}} \left( \sum_{n',m'=1}^{N} B_{n'm'}^{4} \right)_{\text{pde}}.$$  \hspace{1cm} (55)

Each of these two ensemble average/double sum terms is easily shown to have a value of $N^2 \mu^3 + N \mu$, from which fact it follows that

$$S_4 = N^4 \mu^4 + 2N^3 \mu^3 + N^2 \mu^2.$$  \hspace{1cm} (56)

Substituting these result for $S_2$, $S_3$, and $S_4$ into Eq. (51) and simplifying we get the result that

$$\text{Var}\{\hat{C}_{AB}\} = \frac{\mu^2}{N-1} \approx \mu^2 /N,$$  \hspace{1cm} (57)

approximating $N - 1$ on the basis of the presumption that our interest will be in very large values of $N$.

Making use of the result expressed by Eq. (57) along with those presented by Eqs. (42) and (45), we can now write for the signal-to-noise (voltage) ratio, $\text{SNR}_{\mu}$, that is to be associated with the measured or estimated value of the covariance, $\hat{C}_{AB}$, that

$$\text{SNR}_{\mu} = \frac{C_{AB}}{\sqrt{\text{Var}\{\hat{C}_{AB}\}}} = \frac{1}{\sqrt{\text{Var}\{\hat{C}_{AB}\}}} = \frac{1}{\Delta t} \left| F \left( \frac{\theta_{A} - \theta_{B}}{\lambda Z} \right) \right|.$$  \hspace{1cm} (58)

It is appropriate at this point to recall that, taking account of the approximations made earlier, what we really have is only an upper limit on the signal-to-noise (voltage) ratio. Further noting that since the pattern of the target object, $F(r)$, and therefore also the normalized pattern of the target object, $\tilde{F}(r)$, is everywhere nonnegative, then the largest component
of the Fourier transform of the normalized pattern of the target object is the zero spatial frequency component—which, as remarked earlier, is equal to unity—it is convenient to write that

$$\text{SNR}_V < \frac{\pi D^3 (\tilde{\mathcal{P}}/\Delta \nu)}{h \nu} N^{3/2}. \quad (59)$$

It is perhaps worth remarking here that while the value of $\tilde{\mathcal{P}}/\Delta \nu$ is dependent on the mid-band optical frequency, $\nu$, it is essentially independent of the bandwidth, $\Delta \nu$.

With the signal-to-noise ratio result given by Eq. (59) in hand it is appropriate to turn to radiometric considerations and the development of numerical results for the value of the signal-to-noise ratio.

6. SIGNAL-TO-NOISE RATIO

For the development of signal-to-noise ratio results relevant to the ICI concept we will consider a target object at a range $Z = 3.6 \times 10^7$ m (the nominal range to an almost directly overhead geosynchronous satellite), the target object presenting a (more-or-less) circular image pattern—a circular pattern of diameter $\mathcal{D}$. With the objective of setting an upper limit for the value of the signal-to-noise ratio we make the optimistic assumptions (1) that the surface of the target object has a reflectivity of unity, (2) that its solar illumination arrives along a direction that makes only a very small angle with the line of sight along which the telescopes view the target object, and (3) that the solar illumination scattered off the target object is scattered in such a pattern that the target object appears to be a Lambertian source along with the already stated optimistic assumption (4) that the detectors have a quantum efficiency of unity and what has been an implicit assumption (5) that the telescope optics deliver all of the collected optical power to the detectors. This means that since the solar flux density in the vicinity of the Earth (and of the target object) is about $1.4 \, kW/m^2$ then the expected value of the in-band spectral power density, $\tilde{\mathcal{P}}$, at the ground plane has a value of

$$\tilde{\mathcal{P}} = \frac{1.4 \times 10^7 (\frac{\pi \mathcal{D}^2}{4})}{\pi (3.6 \times 10^7)^2} S(\nu)\Delta \nu = 2.70 \times 10^{-13} S(\nu)\Delta \nu, \quad (60)$$

where the function $S(\nu)$ is used here to indicate the spectral distribution of blackbody radiation at a temperature of 5770 K—which is nominally the temperature to be associated with the radiation from the sun. The function $S(\nu)$ has a value equal to the fraction of the optical power that is in a one-Hertz-wide optical frequency band centered at an optical frequency $\nu$. Substituting Eq. (60) into Eq. (59) we get the result that

$$\text{SNR}_V = K(\tilde{\nu})(D\mathcal{D})^2 N^{3/2}, \quad (61)$$

where

$$K(\tilde{\nu}) = \frac{3.19 \times 10^{20} S(\tilde{\nu})}{\tilde{\nu}}. \quad (62)$$

The function $K(\tilde{\nu})$ is shown in Fig. 1. As can be seen the maximum value of $K(\tilde{\nu})$ is only about $2.2 \times 10^{-9}$, and that at a wavelength of about $\lambda = 1.5 \, \mu m$.

The smallness of the value of $K(\tilde{\nu})$ poses a challenge—how can a useful signal-to-noise ratio be achieved? It is to be noted that the signal-to-noise ratio is not directly dependent on the integration time, $\Delta t$, or the spectral bandwidth, $\Delta \nu$. Because of the nature of the phenomenology on which intensity interferometry is based, expanding either of these will bring more photons into each signal-to-noise ratio because it increases the system’s time–bandwidth product and thus reduces the depth of the random modulation of the signal—and it is the correlation of the random modulation that provides the target object image information.

The presence of the factor of $(D\mathcal{D})^2$, the square of the product of the telescope diameter and the target object’s diameter, in the expression for the signal-to-noise ratio given by Eq. (61) calls for some comment. As noted earlier the telescope diameter, $D$, must be so small that the telescope can not resolve even the size, $\mathcal{D}$, of the target object. This imposes the limit that

$$Z(\lambda/D) \ll \mathcal{D}, \text{ or equivalently } D\mathcal{D} \ll \lambda Z. \quad (63)$$

The origin of this limitation arises from the fact that each pair of points in the pair of apertures (one point in each of the two apertures) is contributing to the estimate of a component of the Fourier transform of the target object’s pattern for a spatial frequency that is defined by the separation of that pair of points. If the telescope apertures are large there will be a correspondingly large spread in the separations of different pairs of points. As a consequence the covariance value produced will correspond to an average over a spread of spatial frequencies. This is tolerable only if the spread is over only a set of Fourier transform components whose values are strongly correlated, and they will be strongly correlated only if the frequencies correspond to oscillatory patterns that remain nearly in phase with each other over the limited extent of the target object’s pattern. It is this requirement that leads...
to the formulation of Eq. (63) and the restriction on the allowable value of $D\bar{D}$.

For a wavelength of $\lambda = 1.0 \micron$ the value of $\bar{\lambda}Z$ is 36 m$^2$. For a $D = 3.0$ m size target object the telescope diameter, $D$, would have to be significantly less than 12 m. Considering that the ICI concept calls for developing an image of the target object by inferring the phase for each Fourier transform component from the magnitudes of those components, it would seem that great care needs to be taken to ensure the fundamental soundness of those measured amplitudes—from consideration of which fact we infer that the telescope diameter should be no greater than about $D = 1.0$ m. Accordingly, to achieve even as modest a signal-to-noise ratio as $\text{SNR}_v = 10$ (i.e., 20 dB) would require that $N = 2.55 \times 10^{17}$ sample values. If the integration time were as short as $\Delta t = 1.0 \times 10^{-9}$ s it would require a total time of $T = 2.55 \times 10^8$ s $\equiv$ 70,830 h.

There are two things that have been considered to reduce the required time. One could use a multiplicity of pairs of telescopes. With 100 pairs of telescopes—which number of pairs could be achieved with only 101 properly spaced telescopes—the 70,830 h could be reduced to only 708.3 h. The possibility also exists of working with a multiplicity of closely spaced spectral bands in each telescope—but the spectral bands would have to be very close together or else the measurements would be for significantly different spatial frequency components and so could not be directly combined. If 10 spectral bands were used this would bring down the measurement time by another factor of 10. With 100 pairs of telescopes and 10 spectral bands the required time would come down to about 70.8 h. It is to be noted that the term “properly spaced” means all pairs having the same separation vector—a separation vector corresponding to the spatial frequency being measured. For 101 telescopes to provide 100 properly spaced pairs of telescopes all of the telescopes would have to be placed, uniformly spaced, on a straight line.

But then one has to make allowance for the fact that covariance values would have to be developed for many different telescope spacings, that is, for many different components of the target object’s Fourier transform—probably for about as many components as there are to be resolution cells (pixels) on the recovered image of the target object. It is hard to see how anything useful could be produced with less than 10 (and more likely any thing less than 100) components. This would seem to imply a total measurement period of the order of 700–7000 h of data collection time. The increase in the total measurement time need not actually be as large as the number of spatial frequencies that have to be measured would seem to imply. This is because the same telescope output can be used in a multiplicity of pairings—each pairing being with a different telescope and thus for a different separation and spatial frequency. Of course not all of the pairings would correspond to separations and spatial frequencies of interest, but a large number would—and accordingly the total measurement time would be a some what more modest multiple of the 70.8 h figure. The exact multiple would depend on operational and scheduling considerations that are beyond the scope of this paper.

Considering the unavoidable down times (for day/night, unfavorable weather, and the unavoidable limited reliability of hardware) it is hard to see how ICI could be formulated so as to provide a responsive capability.

ACKNOWLEDGMENTS

The authors would like to acknowledge Travis W. Axtell for his assistance on typesetting the paper and figure for the journal template. Approved for public release; distribution is unlimited. The views expressed in this document are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

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