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IEEE

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Ziomek, L. J. "Three-dimensional ray acoustics: new expressions for the amplitude, eikonal, and phase functions." IEEE Journal of Oceanic Engineering 14.4 (1989): 396-399.  
<https://hdl.handle.net/10945/59967>

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# Three-Dimensional Ray Acoustics: New Expressions for the Amplitude, Eikonal, and Phase Functions

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**Abstract**—New three-dimensional ray acoustics' expressions for the amplitude, eikonal, and phase along a ray path are derived.

**Keywords**—Three-dimensional ray acoustics, amplitude, eikonal, and phase functions.

## I. INTRODUCTION

THE main purpose of this paper is to derive new expressions for the amplitude, eikonal, and phase along a ray path based on three-dimensional ray acoustics. The new expressions clearly indicate the numerical calculations that must be performed in order to evaluate these functions. The ocean medium is characterized by a three-dimensional random index of refraction which is decomposed into deterministic and random components.

## II. ANALYSIS

The propagation of small-amplitude acoustic signals in the ocean can be described by the following linear, homogeneous wave equation:

$$\nabla^2 \varphi(t, \mathbf{r}) - \frac{1}{c^2(\mathbf{r})} \frac{\partial^2}{\partial t^2} \varphi(t, \mathbf{r}) = 0 \quad (1)$$

where  $\varphi(t, \mathbf{r})$  is the velocity potential in square meters per second at time  $t$  and position  $\mathbf{r} = (x, y, z)$ , and  $c(\mathbf{r})$  is the speed of sound in the ocean in meters per second. If we assume a time-harmonic dependence for the velocity potential, that is, if

$$\varphi(t, \mathbf{r}) = \varphi(\mathbf{r}) \exp(+j2\pi ft) \quad (2)$$

then substituting (2) into (1) yields the following linear, homogeneous, time-independent Helmholtz wave equation:

$$\nabla^2 \varphi(\mathbf{r}) + k_0^2 n^2(\mathbf{r}) \varphi(\mathbf{r}) = 0 \quad (3)$$

where

$$k_0 = 2\pi f / c_0 = 2\pi / \lambda_0 \quad (4)$$

is the constant, reference wave number in radians per meter,

$$n(\mathbf{r}) = c_0 / c(\mathbf{r}) \quad (5)$$

is the random, three-dimensional dimensionless index of refraction, and

$$c_0 = c(\mathbf{r}_0) = f\lambda_0 \quad (6)$$

Manuscript received January 13, 1988; revised May 9, 1989. This work was supported by the ONR and the Naval Postgraduate School.

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IEEE Log Number 8929729.

is the constant, reference speed of sound at the source position  $\mathbf{r}_0 = (x_0, y_0, z_0)$ . Note that the wave number

$$k(\mathbf{r}) = 2\pi f / c(\mathbf{r}) \quad (7)$$

can be expressed as

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) \quad (8)$$

and that  $k(\mathbf{r}_0) = k_0$  since  $n(\mathbf{r}_0) = 1$ .

The index of refraction is commonly written as [1], [2]

$$n(\mathbf{r}) = n_D(\mathbf{r}) + n_R(\mathbf{r}) \quad (9)$$

or

$$n(\mathbf{r}) = n_D(\mathbf{r}) + \sigma(\mathbf{r})n_{NR}(\mathbf{r}) \quad (10)$$

where  $n_D(\mathbf{r})$  is the deterministic component and is sometimes referred to as the deterministic or background sound channel,  $n_R(\mathbf{r})$  is the random, zero-mean component,  $\sigma(\mathbf{r})$  is the standard deviation of  $n_R(\mathbf{r})$ , and

$$n_{NR}(\mathbf{r}) = n_R(\mathbf{r}) / \sigma(\mathbf{r}) \quad (11)$$

is the normalized random component with zero mean and variance equal to unity. Note that the expected (average) value of  $n(\mathbf{r})$  is equal to  $n_D(\mathbf{r})$ .

An approximate solution of the Helmholtz wave equation given by (3), based on the method of three-dimensional ray acoustics, is given by

$$\varphi(\mathbf{r}) = a(\mathbf{r}) \exp[-jk_0 W(\mathbf{r})] = a(\mathbf{r}) \exp[+j\theta(\mathbf{r})] \quad (12)$$

where

$$a(\mathbf{r}) = a(\mathbf{r}_0) \left[ \frac{S_0}{n(\mathbf{r})S} \right]^{1/2} \quad (13)$$

is the real amplitude function,  $a(\mathbf{r}_0)$  is the amplitude at the source, and  $S_0$  and  $S$  are the very small wavefront surface areas at the ends of the ray-tube segment specified by the position vectors  $\mathbf{r}_0$  and  $\mathbf{r}$ , respectively [3]–[5]. In addition,

$$W(\mathbf{r}) = W(\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} n(x, y, z) ds \quad (14)$$

is the eikonal in meters,  $W(\mathbf{r}_0)$  is some initial value of the eikonal at  $\mathbf{r}_0$ , and  $s$  is the arc length in meters [6], [7]. And finally,

$$\theta(\mathbf{r}) = -k_0 W(\mathbf{r}) = \theta(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} k(x, y, z) ds \quad (15)$$

is the real phase function in radians,  $\theta(\mathbf{r}_0)$  is some initial value of the phase at  $\mathbf{r}_0$ , and where it is understood that  $x = x(s)$ ,  $y = y(s)$ , and  $z = z(s)$  in the arguments of the index of refraction and the wave number in (14) and (15), respectively [7], [8]. With respect to wave propagation in random media, the method of ray acoustics is valid only for problems involving weak scatter and when the scale sizes of the medium are large compared with the sound wavelength [2], [9].

Equations (13) through (15) are the most common three-dimensional ray acoustics' expressions for the amplitude, eikonal, and phase, respectively. The disadvantage of these common expressions is that there is no additional information given on how to actually solve for the cross-sectional areas  $S_0$  and  $S$  in (13) and how to evaluate the integrals appearing in (14) and (15). However, *new* expressions for these quantities can be derived that clearly indicate the numerical calculations that must be performed.

*A. The Amplitude Function*

An alternate, new expression for the amplitude  $a(\mathbf{r})$  along a ray path can be obtained as follows. We begin with the transport equation as given by [10]

$$\frac{d}{ds} \ln a(\mathbf{r}) = -\frac{\nabla^2 W(\mathbf{r})}{2n(\mathbf{r})}. \tag{16}$$

The solution of (16) can be expressed as

$$\ln \left[ \frac{a(\mathbf{r})}{a(\mathbf{r}_0)} \right] = -\frac{1}{2} \int_{r_0}^r \frac{\nabla^2 W(\mathbf{r})}{n(\mathbf{r})} ds \tag{17}$$

and, as a result [8], [11],

$$a(\mathbf{r}) = a(\mathbf{r}_0) \exp \left[ -\frac{1}{2} \int_{r_0}^r \frac{\nabla^2 W(\mathbf{r})}{n(\mathbf{r})} ds \right]. \tag{18}$$

Equation (18) can be simplified further.

Since [6], [12]

$$\nabla W(\mathbf{r}) = n(\mathbf{r})\hat{n}(\mathbf{r}) \tag{19}$$

and

$$\nabla^2 W(\mathbf{r}) = \nabla \cdot \nabla W(\mathbf{r}) \tag{20}$$

substituting (19) into (20) yields [6]

$$\nabla^2 W(\mathbf{r}) = \nabla \cdot n(\mathbf{r})\hat{n}(\mathbf{r}) = n(\mathbf{r})\nabla \cdot \hat{n}(\mathbf{r}) + \hat{n}(\mathbf{r}) \cdot \nabla n(\mathbf{r}) \tag{21}$$

where

$$\hat{n}(\mathbf{r}) = u(\mathbf{r})\hat{x} + v(\mathbf{r})\hat{y} + w(\mathbf{r})\hat{z} \tag{22}$$

is the unit vector in the direction of  $\nabla W(\mathbf{r})$  and  $u(\mathbf{r})$ ,  $v(\mathbf{r})$ , and  $w(\mathbf{r})$  are the direction cosines with respect to the  $X$ ,  $Y$ , and  $Z$  axes, respectively. And, since [6]

$$\frac{d}{ds} n(\mathbf{r}) = \nabla n(\mathbf{r}) \cdot \hat{n}(\mathbf{r}) \tag{23}$$

substituting (23) into (21) yields

$$\frac{\nabla^2 W(\mathbf{r})}{n(\mathbf{r})} = \nabla \cdot \hat{n}(\mathbf{r}) + \frac{1}{n(\mathbf{r})} \frac{d}{ds} n(\mathbf{r}). \tag{24}$$

Substituting (24) into (18) yields

$$a(\mathbf{r}) = a(\mathbf{r}_0) \exp \left[ -\frac{1}{2} \int_{n(\mathbf{r}_0)}^{n(\mathbf{r})} \frac{1}{n(\mathbf{r})} dn(\mathbf{r}) \right] \cdot \exp \left[ -\frac{1}{2} \int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds \right] \tag{25}$$

$$= a(\mathbf{r}_0) \exp [\ln |n(\mathbf{r})/n(\mathbf{r}_0)|^{-1/2}] \cdot \exp \left[ -\frac{1}{2} \int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds \right] \tag{26}$$

and, noting that the real index of refraction is always positive and that  $n(\mathbf{r}_0) = 1$ , we obtain the following expression for the amplitude along a ray path:

$$a(\mathbf{r}) = a(\mathbf{r}_0) \left[ \frac{1}{n(\mathbf{r})} \right]^{1/2} \exp \left[ -\frac{1}{2} \int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds \right] \tag{27}$$

or, upon substituting (5) into (27),

$$a(\mathbf{r}) = a(\mathbf{r}_0) \left[ \frac{c(\mathbf{r})}{c(\mathbf{r}_0)} \right]^{1/2} \exp \left[ -\frac{1}{2} \int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds \right]. \tag{28}$$

Next, in order to make (27) and (28) more amenable to numerical calculations, we must try to simplify the integral

$$\int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds.$$

Since (see (22))

$$\nabla \cdot \hat{n}(\mathbf{r}) ds = \frac{\partial}{\partial x} u(\mathbf{r}) ds + \frac{\partial}{\partial y} v(\mathbf{r}) ds + \frac{\partial}{\partial z} w(\mathbf{r}) ds \tag{29}$$

and [13]

$$ds = dx/u(\mathbf{r}) \tag{30}$$

$$ds = dy/v(\mathbf{r}) \tag{31}$$

and

$$ds = dz/w(\mathbf{r}) \tag{32}$$

substituting (30) through (32) into (29) yields

$$\nabla \cdot \hat{n}(\mathbf{r}) ds = \frac{1}{u(\mathbf{r})} \frac{\partial}{\partial x} u(\mathbf{r}) dx + \frac{1}{v(\mathbf{r})} \frac{\partial}{\partial y} v(\mathbf{r}) dy + \frac{1}{w(\mathbf{r})} \frac{\partial}{\partial z} w(\mathbf{r}) dz. \tag{33}$$

Therefore, with the use of (33),

$$\begin{aligned} \int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds = & \int_{x_0}^x \frac{1}{u(\zeta, y, z)} \frac{\partial}{\partial \zeta} u(\zeta, y, z) d\zeta \\ & + \int_{y_0}^y \frac{1}{v(x, \zeta, z)} \frac{\partial}{\partial \zeta} v(x, \zeta, z) d\zeta \\ & + \int_{z_0}^z \frac{1}{w(x, y, \zeta)} \frac{\partial}{\partial \zeta} w(x, y, \zeta) d\zeta \end{aligned} \quad (34)$$

which is the desired result. Therefore, either (27) or (28), in conjunction with (34), represent *new* expressions for the amplitude along a ray path. By comparing (13) and (27), it can be seen that the task of computing  $S_0$  and  $S$  is equivalent to evaluating the integral of the divergence of the unit vector along a ray path; that is,

$$\left[ \frac{S_0}{S} \right]^{1/2} \equiv \exp \left[ -\frac{1}{2} \int_{r_0}^r \nabla \cdot \hat{n}(\mathbf{r}) ds \right] \quad (35)$$

where the integral is given by (34). It can be shown that when the index of refraction (speed of sound) is an arbitrary function of depth  $y$  only, the new amplitude function, which was derived based on three-dimensional ray acoustics, reduces to the amplitude function that one obtains via the WKB method; that is,

$$a(\mathbf{r}) = a(y) = A / |k_y(y)|^{1/2} \quad (36)$$

where

$$A = a(\mathbf{r}_0) |k_y(y_0)|^{1/2} \quad (37)$$

and  $k_y(y)$  is the propagation vector component in the  $Y$  direction [14], [15].

### B. The Eikonal and Phase Functions

An alternate, new expression for the eikonal can be obtained as follows. We begin by computing the directional derivative of the eikonal; that is,

$$\frac{d}{ds} W(\mathbf{r}) = \frac{\partial}{\partial x} W(\mathbf{r}) \frac{dx}{ds} + \frac{\partial}{\partial y} W(\mathbf{r}) \frac{dy}{ds} + \frac{\partial}{\partial z} W(\mathbf{r}) \frac{dz}{ds} .$$

Since [13], [16]

$$\frac{\partial}{\partial x} W(\mathbf{r}) = n(\mathbf{r}) u(\mathbf{r}) \quad (39)$$

$$\frac{\partial}{\partial y} W(\mathbf{r}) = n(\mathbf{r}) v(\mathbf{r}) \quad (40)$$

and

$$\frac{\partial}{\partial z} W(\mathbf{r}) = n(\mathbf{r}) w(\mathbf{r}) \quad (41)$$

substituting (39) through (41) into the right-hand side of (38)

yields

$$\frac{d}{ds} W(\mathbf{r}) = n(\mathbf{r}) u(\mathbf{r}) \frac{dx}{ds} + n(\mathbf{r}) v(\mathbf{r}) \frac{dy}{ds} + n(\mathbf{r}) w(\mathbf{r}) \frac{dz}{ds} \quad (42)$$

with solution

$$\begin{aligned} W(\mathbf{r}) = & W(\mathbf{r}_0) + \int_{x_0}^x n(\zeta, y, z) u(\zeta, y, z) d\zeta \\ & + \int_{y_0}^y n(x, \zeta, z) v(x, \zeta, z) d\zeta \\ & + \int_{z_0}^z n(x, y, \zeta) w(x, y, \zeta) d\zeta . \end{aligned} \quad (43)$$

Equation (43) is the *new* expression for the eikonal and, since

$$\theta(\mathbf{r}) = -k_0 W(\mathbf{r}) \quad (44)$$

substituting (43) into (44) and using (8) yields the following *new* expression for the phase function:

$$\begin{aligned} \theta(\mathbf{r}) = & \theta(\mathbf{r}_0) - \int_{x_0}^x k_x(\zeta, y, z) d\zeta - \int_{y_0}^y k_y(x, \zeta, z) d\zeta \\ & - \int_{z_0}^z k_z(x, y, \zeta) d\zeta \end{aligned} \quad (45)$$

where

$$k_x(\mathbf{r}) = k(\mathbf{r}) u(\mathbf{r}) \quad (46)$$

$$k_y(\mathbf{r}) = k(\mathbf{r}) v(\mathbf{r}) \quad (47)$$

and

$$k_z(\mathbf{r}) = k(\mathbf{r}) w(\mathbf{r}) \quad (48)$$

are the propagation vector components in the  $X$ ,  $Y$ , and  $Z$  directions, respectively. When the index of refraction (speed of sound) is an arbitrary function of depth  $y$  only, the propagation vector components  $k_x(\mathbf{r}) = k_x$  and  $k_z(\mathbf{r}) = k_z$  are *constants* and  $k_y(\mathbf{r}) = k_y(y)$  [15]. As a result, (45) reduces to

$$\begin{aligned} \theta(x, y, z) = & \theta(x_0, y_0, z_0) - k_x(x - x_0) \\ & - \int_{y_0}^y k_y(\zeta) d\zeta - k_z(z - z_0). \end{aligned} \quad (49)$$

The phase integral appearing in (49) is identical with the phase integral obtained via the WKB method [14], [15].

In order to determine the amplitude along a ray path according to (28) and (34), and in order to determine the phase along a ray path according to (45) through (48), the direction cosines  $u(\mathbf{r})$ ,  $v(\mathbf{r})$ , and  $w(\mathbf{r})$  must first be determined by solving the ray equations [17], [18]. This is not necessarily a shortcoming, since in order to draw ray diagrams for a three-dimensional sound-speed profile, the ray equations must be solved anyway. And, finally, note that if the index of

refraction is random, the direction cosines will be random as well. However, for problems dealing with wave propagation in a random, inhomogeneous medium, amplitude and phase calculations are performed by carrying out integrations along *deterministic unperturbed ray paths* [19], that is, using the deterministic components of the direction cosines in (34), (43), and (45) through (48).

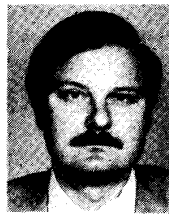
## ACKNOWLEDGMENT

Discussions held with Prof. S. M. Flatté of the University of California at Santa Cruz and several of his graduate students, and with Prof. J. H. Miller of the Naval Postgraduate School regarding the content of an earlier version of this paper, were very helpful and very much appreciated.

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