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Lead Time Demand Modeling in Continuous Review Supply Chain Models

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Preface & Acknowledgements

Welcome to our Tenth Annual Acquisition Research Symposium! We regret that this year it will be a “paper only” event. The double whammy of sequestration and a continuing resolution, with the attendant restrictions on travel and conferences, created too much uncertainty to properly stage the event. We will miss the dialogue with our acquisition colleagues and the opportunity for all our researchers to present their work. However, we intend to simulate the symposium as best we can, and these *Proceedings* present an opportunity for the papers to be published just as if they had been delivered. In any case, we will have a rich store of papers to draw from for next year’s event scheduled for May 14–15, 2014!

Despite these temporary setbacks, our Acquisition Research Program (ARP) here at the Naval Postgraduate School (NPS) continues at a normal pace. Since the ARP’s founding in 2003, over 1,200 original research reports have been added to the acquisition body of knowledge. We continue to add to that library, located online at www.acquisitionresearch.net, at a rate of roughly 140 reports per year. This activity has engaged researchers at over 70 universities and other institutions, greatly enhancing the diversity of thought brought to bear on the business activities of the DoD.

We generate this level of activity in three ways. First, we solicit research topics from academia and other institutions through an annual Broad Agency Announcement, sponsored by the USD(AT&L). Second, we issue an annual internal call for proposals to seek NPS faculty research supporting the interests of our program sponsors. Finally, we serve as a “broker” to market specific research topics identified by our sponsors to NPS graduate students. This three-pronged approach provides for a rich and broad diversity of scholarly rigor mixed with a good blend of practitioner experience in the field of acquisition. We are grateful to those of you who have contributed to our research program in the past and encourage your future participation.

Unfortunately, what will be missing this year is the active participation and networking that has been the hallmark of previous symposia. By purposely limiting attendance to 350 people, we encourage just that. This forum remains unique in its effort to bring scholars and practitioners together around acquisition research that is both relevant in application and rigorous in method. It provides the opportunity to interact with many top DoD acquisition officials and acquisition researchers. We encourage dialogue both in the formal panel sessions and in the many opportunities we make available at meals, breaks, and the day-ending socials. Many of our researchers use these occasions to establish new teaming arrangements for future research work. Despite the fact that we will not be gathered together to reap the above-listed benefits, the ARP will endeavor to stimulate this dialogue through various means throughout the year as we interact with our researchers and DoD officials.

Affordability remains a major focus in the DoD acquisition world and will no doubt get even more attention as the sequestration outcomes unfold. It is a central tenet of the DoD’s Better Buying Power initiatives, which continue to evolve as the DoD finds which of them work and which do not. This suggests that research with a focus on affordability will be of great interest to the DoD leadership in the year to come. Whether you’re a practitioner or scholar, we invite you to participate in that research.

We gratefully acknowledge the ongoing support and leadership of our sponsors, whose foresight and vision have assured the continuing success of the ARP:

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Lead Time Demand Modeling in Continuous Review Supply Chain Models

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Abstract

This paper introduces a mixture distribution approach to modeling the probability density function for lead time demand (LTD) in problems where a continuous review inventory system is implemented. The method differs from the typical “moment-matching” approach by focusing on building up an accurate, closed-form approximation to the LTD distribution from its components by using mixtures of truncated exponential (MTE) functions. First, construction of the LTD is illustrated and the approach is compared to two other possible LTDs. This distribution is then utilized to determine optimal order policies in cases where a buyer makes its decisions alone, and later in a situation where members of a two-level supply chain coordinate their actions.

Introduction

Numerous probability models have been suggested for representing uncertain demand during lead time (LT) in continuous-review inventory management systems when both LT and demand per unit time (DPUT) are variable. A common approach to finding a distribution for lead time demand (LTD) involves modeling LT and DPUT with standard probability density functions (PDFs). Based on the distributions assigned, a compound probability distribution is determined for demand during lead time, or LTD. The latter distribution is used to determine reorder point and safety stock policies, and may be used to estimate inventory costs. In some cases, analytical formulas for optimal reorder point, safety stock, or stockout costs are available in terms of the compound distribution’s parameters, while in other situations the values associated with certain percentiles of the compound LTD distribution are estimated to provide these values. Although the problem of finding an appropriate LTD distribution has been well studied, papers written in recent years have continued to pursue methods that overcome unrealistic distributional assumptions (Ruiz-Torres & Mahmoodi, 2010; Vernimmen, Dullaert, Willimé, & Witlox, 2008).

This paper illustrates an approach for constructing a mixture distribution for LTD that allows the LT and DPUT distributions to be state-dependent. This method also allows input distributions that take any standard or empirical form. Use of the mixture distribution technique is first demonstrated in the context described by Cobb (2013), which is a single-item continuous-review inventory model for one buyer. For single-firm operating in a continuous-review inventory system, the mixture distribution method for modeling the LTD distribution differs from the typical “moment-matching” approach. The method focuses on building up an accurate, closed-form approximation to the LTD distribution from its components by using mixtures of truncated exponential (MTE) functions.

After the mixture distribution approach is described, a two-level supply chain model where the buyer operates under uncertain demand and utilizes a continuous review inventory system will be considered. In this two-echelon supply chain model, credit terms (Chaharsooghi & Heydari, 2010), quantity discounts (Li & Liu, 2006; Chaharsooghi, Heydari, & Kamalabadi, 2011), and rebates (Cobb & Johnson, 2013) have been suggested as coordinating incentives that allow the supply chain members to divide the cost savings resulting from coordinating their order quantity and reorder point decisions. In each of these cases, LTD is assumed to be normally distributed. This assumption is not always realistic, particularly when DPUT and LT are each random variables such that LTD has a compound probability distribution (Eppen & Martin, 1988; Lau & Lau, 2003; Lin, 2008). This paper will incorporate the previously described model (Cobb, 2013) into the two-echelon supply chain problem to show that this model can obviate the need to assume that demand for the entire LT period is normally distributed.

The next section describes LTD distributions and uses an example dataset to show how standard PDFs can be used as approximations to the LTD distribution. The mixture distribution method is also used for the example problem. Next, the different approximations to the LTD distribution are used to find optimal inventory order quantity and reorder point policies. This is followed by an illustration of how the mixture distribution approach can allow more complicated LTD distributions to be incorporated into such problems. The two-level supply chain model is then introduced, and the mixture distribution approach is used to model LTD in the context of decentralized, centralized, and coordinated supply chains. The final section concludes the paper.

Lead Time Demand Distributions

LTD in a continuous-review inventory system is often assumed to follow a compound probability distribution. Suppose *L* is a random variable for lead time (LT) and *D* represents random demand per unit of time (DPUT). LTD is a random variable *X* determined as

$$X = D_1 + D_2 + \cdots + D_i + \cdots + D_L \ . \tag{1}$$

Therefore, *X* is a sum of random, independent, and identically distributed (i.i.d.) instances of demand. The mean and variance of *X* can be calculated as

$$E(X) = E(L) \cdot E(D) \text{ and } Var(X) = E(L) \cdot Var(D) + [E(D)]^2 \cdot Var(L). \tag{2}$$

Suppose the data in Table 1 is available to estimate an LTD distribution. This table contains 50 observations of daily demand for an inventory item and 10 observations for LT on orders of the same item. The expected value of daily demand is *E*(*D*)=2.88, and the variance of this random variable is *Var*(*D*)=2.84. LT has an expected value and variance of *E*(*L*)=5.3 and *Var*(*L*)=6.9, respectively. According to the formulas in Equation 2, the expected value and variance of LTD are *E*(*X*)=15.26 and *Var*(*X*)=72.3, respectively.

The remainder of this section will illustrate three possible methods for approximating the LTD distribution underlying the data in Table 1.

Normal Approximation

The service level is defined as the percentage of replenishment order cycles where demand during LT is satisfied. To determine the reorder point (*R*) required to achieve a desired service level, a typical textbook approach is to assume the LTD distribution is normal and use normal distribution tables or Excel formulas. For example, to find the *R* needed to achieve a 95% service level for the LTD distribution with expected value and variance described in Table 1, the Excel formula NORM.INV(0.95,15.26,72.3^0.5) can be used to find *R* = 29.25.

Table 1. Observations for Daily Demand and Lead Time

Daily demand (DPUT)	1	2	2	1	4	1	1	1	1	1
	3	5	3	2	5	4	2	2	3	2
	2	3	3	3	1	3	6	3	6	2
	5	1	5	3	2	6	1	2	4	1
	3	2	2	2	6	5	5	1	3	7
Lead time (LT)	3	5	3	4	4	5	5	10	5	10

The normal approximation to the LTD distribution and the reorder point *R*=29.25 are illustrated graphically in Figure 1. By implementing this policy, we would expect to stockout on 5% of replenishment order cycles.

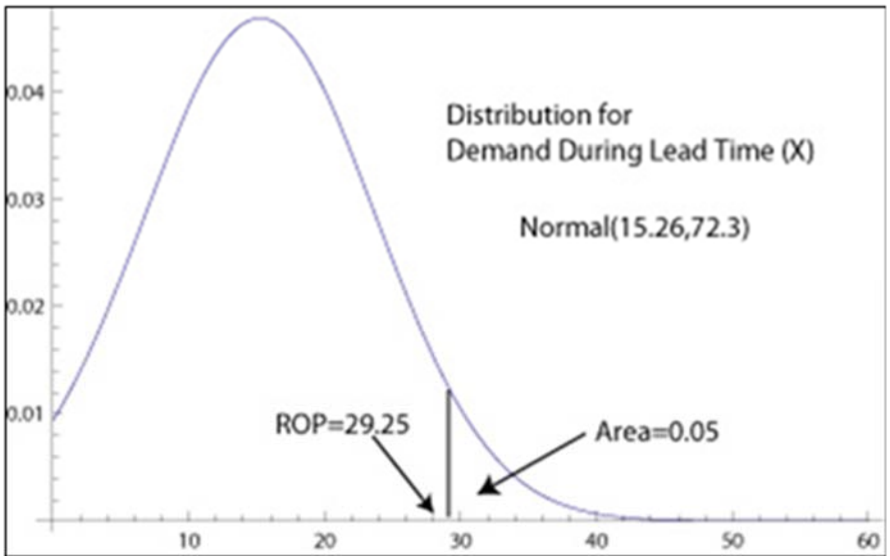

Figure 1. LTD Distribution and Reorder Point

Negative Binomial Approximation

Although the normal approximation to the LTD distribution is popular, there are numerous other approximations that have been suggested in the literature. For example, Taylor (1961) suggested using the negative binomial (NB) distribution for the case where the Poisson distribution is a good fit for DPUT and LT has a gamma distribution. We denote the approximate LTD distribution by $\hat{f}$. Here we assume the NB(*r*,*p*) distribution for LTD is

$$\hat{f}(x;r,p) = \frac{\Gamma(x+r)}{x!\cdot\Gamma(r)}(1-p)^r p^x \quad x = 0,1,2,\ldots \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. Given this formulation, *E*(*X*)=*rp*/(1-*p*) and *Var*(*X*)=*E*(*X*)/(1-*p*). There are two ways of finding a reorder point that will provide an appropriate service level with this NB formulation. Taylor (1961) provided a formula to calculate stockout probabilities as a function of the underlying Poisson and gamma distributions. These can be calculated for possible reorder point values until a suitable value that meets the service level objective is found. Excel can also be used to enumerate the probabilities of achieving a certain service level with various possible values of *R*. Unfortunately, the built-in NEGBINOM.DIST function only accepts integer values of the *r* parameter, so these probabilities must be calculated using the formula in Equation 3 and the GAMMALN function.

For the data in Table 1, we can use the empirical expected value and variance to solve two equations and two unknowns and obtain *r* = 4.08 and *p* = 0.79. This NB distribution is shown in Figure 2. The value of *R* that provides approximately a 95% service level is *R* = 31.

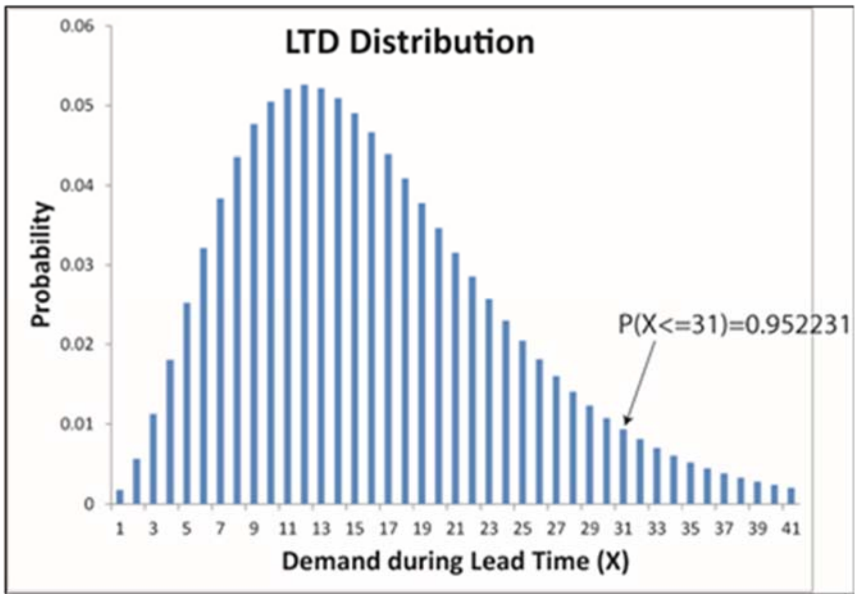

Figure 2. Negative Binomial Distribution for Lead Time Demand

This solution is essentially the same as the one found using Taylor's (1961) analytical formulas. In this case, the Poisson daily demand assumption may be reasonable because *E*(*D*) and *Var*(*D*) are very similar, a feature of the Poisson distribution.

Mixtures of Truncated Exponentials Approximation

The functional form of some PDFs, such as the negative binomial PDF in Equation 3, does not permit integration in closed-form. The means that the result of an expected value calculation with such a PDF does not have a functional form that can be used for further computation. These calculations could include, for example, building a cost function to perform nonlinear optimization to find optimal inventory policies. One approach suggested to overcome this limitation is the MTE model (Moral, Rumí, & Salmerón, 2001).

An example of a four-piece, two-term (ignoring the constant) MTE function that can be used to model LTD given an LT of *L* = 3 for the problem in the previous section is

$$\hat{f}_{X|\{L=3\}}(x) = \begin{cases} -0.7148 + 0.6681\exp\{0.0325x\} + 0.000048\exp\{0.989x\} & \text{if } 2.5 \le x < 5 \\ -96.721 - 318.54\exp\{-1.945x\} + 96.76\exp\{0.000128x\} & \text{if } 5 \le x < 8 \\ 0.1383 - (1.63E-06)\exp\{x\} + (2.89E-09)\exp\{1.5x\} & \text{if } 8 \le x < 11.5 \\ -0.0252 + 0.9786\exp\{-0.205x\} & \text{if } 11.5 \le x \le 17.5 \end{cases} \quad (4)$$

This function was found by simulating 500 series of three observations for daily demand from values in Table 1 using a bootstrapping approach. The constants—coefficients on the exponential terms and coefficients on the variable *X*—were determined by fitting a function to the simulated histogram. There is an established literature on fitting MTE functions to historical data; in this case, the method suggested by Moral et al. (2002) was utilized. A graphical view of the MTE function overlaid on the simulated histogram is shown in Figure 3.

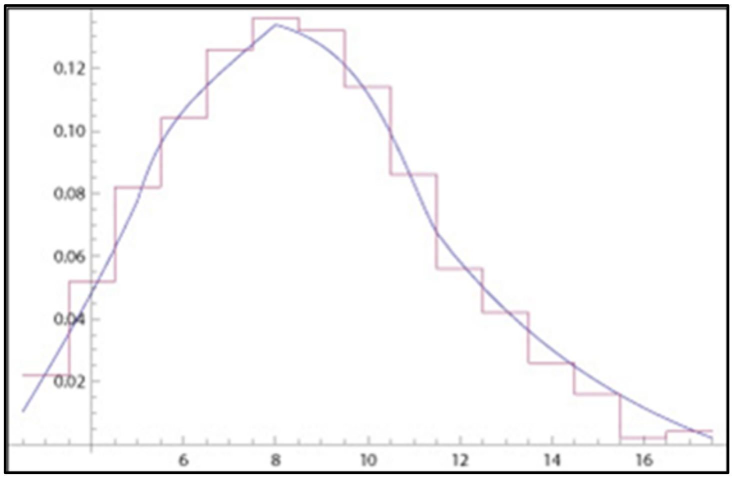

Figure 3. Mixtures of Truncated Exponentials Distribution for Lead Time Demand Given a Lead Time of Three Days

Similar functions $\hat{f}_{X|\{L=l\}}$ can be constructed for the other possible LT values, *L* = 4, 5, and 10. From the data on LT observations in Table 1, we can estimate *P*(*L*=3) = *P*(*L*=4) = *P*(*L*=10) = 0.2 and *P*(*L*=5) = 0.4. A mixture distribution approach (Cobb, 2013) can be employed to find the LTD distribution. Here, the LTD distribution is determined as

$$\hat{f}_X(x) = P(L=3)\cdot\hat{f}_{X|\{L=3\}}(x) + P(L=4)\cdot\hat{f}_{X|\{L=4\}}(x) + P(L=5)\cdot\hat{f}_{X|\{L=5\}}(x) + P(L=10)\cdot\hat{f}_{X|\{L=10\}}(x). \quad (5)$$

The MTE function is shown in Figure 4, overlaid on the previously described NB distribution. This MTE function has 17 pieces and up to six terms in each piece. For illustrative purposes, a continuous NB parameterization is displayed. Because the class of MTE functions is closed under addition, multiplication, and integration (Moral et al., 2001), the mixture distribution resulting from the calculation above is also an MTE function. Thus, it retains the same desirable mathematical properties.

We can perform closed-form integrations of the MTE LTD distribution to find a reorder point that achieves a desired service level. In this case,

$$\int_0^{33.3} \hat{f}_X(x)\,dx \approx 0.95\,, \quad (6)$$

so we can set *R* = 33.3 to obtain a 95% service level.

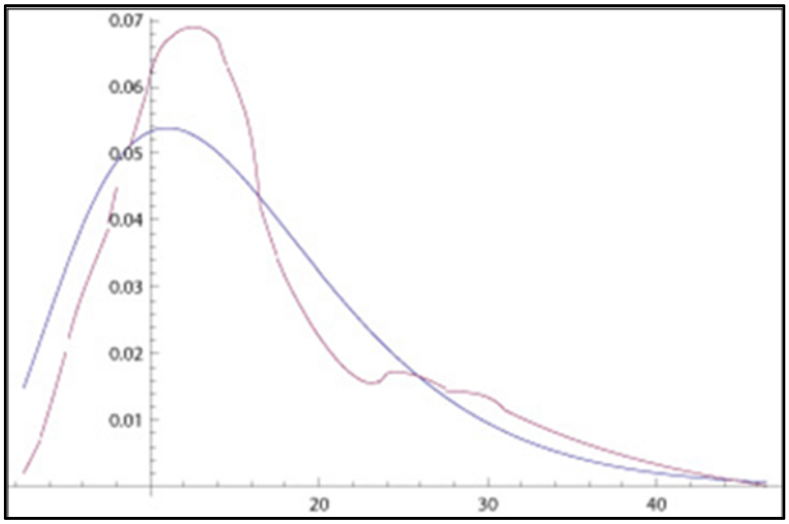

Figure 4. Mixtures of Truncated Exponentials Lead Time Demand Distribution Overlaid on a Negative Binomial Approximation

The next section discusses the use of the MTE function for finding inventory policies in a continuous-review inventory system.

Finding Inventory Policies

Suppose that we want to determine an optimal order quantity and reorder point in a continuous-review inventory system (a “(*Q,R*)” policy). We consider four models that could be used to find the best policy given the data available (see Table 1): (1) a normal approximation to the LTD distribution; (2) the NB approximation to the LTD distribution; (3) the MTE mixture distribution; and (4) a simulation-optimization model that simulates LT and LTD values from the empirical distributions developed from Table 1. We term the latter model the “actual” solution.

A simple cost function with no backordering allowed (Johnson & Montgomery, 1974) for this problem is

$$TC(Q,R) = K \cdot \frac{Y}{Q} + \frac{\pi \cdot Y \cdot S_R}{Q} + h \cdot (0.5Q + R - E(X))\ . \quad (7)$$

In this equation, *K* is the fixed cost per order, *Y* is the expected annual demand, *h* is the holding cost per unit per year, and π is the stockout cost per unit. The average inventory includes safety stock of *R*-*E*(*X*). The shape of the distribution for LTD determines the expected shortage per cycle, S_R. For a given reorder point,

$$S_R = \int_R^{\infty} (x - R) \cdot \hat{f}_X(x)\, dx. \quad (8)$$

Suppose $Y=E(D) \cdot 250$ working days = 720, *K*=30, *h*=4, and π=5. The key to finding an optimal (*Q,R*) combination is to evaluate S_R as part of constructing the total cost function in Equation 7. With the MTE function, the calculation in Equation 8 can be performed in closed-form, and the result substituted into Equation 7 to obtain a closed-form total cost function. The expected shortage per cycle as a function of *R* is an eight-piece expression, with selected terms shown below:

$$\hat{S}_R(r) = \begin{cases} -3876.5 + 4.66\exp\{-0.205r\} + 6.31\exp\{-0.172r\} \\ +3888.1\exp\{0.005r\} - 21.82r - 0.04r^2 & \text{if } 16.15 \le r < 16.5 \\ -3890.6 + 4.66\exp\{-0.205r\} + 6.31\exp\{-0.172r\} \\ +3888.1\exp\{0.005r\} + 20.6\exp\{-0.140r\} - 20.64r - 0.07r^2 & \text{if } 16.5 \le r < 17.5 \\ -3889.2 + 6.31\exp\{-0.172r\} + 3888.1\exp\{0.005r\} \\ +20.6\exp\{-0.140r\} - 20.76r - 0.07r^2 & \text{if } 17.5 \le r < 23.5 \\ & \vdots \\ -8.74 + 29.87\exp\{-0.78r\} + 0.28r - 0.002r^2 & \text{if } 31 \le r < 46.5\,. \end{cases} \tag{9}$$

Optimization over the resulting cost function is fast. The example here was solved in Mathematica 9.0 by using the ArgMin function. The results obtained using the four methods under consideration are shown in Table 2. An iterative approach (Hadley & Whitin, 1963) in combination with numerical integration was implemented to find the solutions using the normal or NB approximations. The table shows the values *Q** and *R** which—when implemented simultaneously—minimize annual total cost. The computing (CPU) times required to obtain the solutions are also shown. The simulation-optimization solution was simply stopped after running for several hours, and the values obtained were assumed to be the best possible solution.

Table 2. Results for Inventory Policies Determined Using Four Approaches

Method	*Q**	*R**	*TC*	CPU (sec.)
Normal Approximation	108	25	482.99	3.57
NB Approximation	110	25	482.89	3.76
MTE Mixture Distribution	110	27	481.10	1.26
Simulation-Optimization	108	27	480.82	∞

Table 2 shows that the MTE mixture distribution works equally as well as the other approaches when implemented to obtain an optimal (*Q*,*R*) policy. The next section illustrates that the mixture distribution approach can be used to model more complicated LTD distributions.

State-Dependent Variables

The advantage of the mixture distribution approach (Cobb, 2013) in inventory management problems is that more complex LTD distributions can be constructed by building the model from its components while still maintaining a closed-form representation. In some cases, expert knowledge can be used to assign state-dependent distributions for DPUT and/or LT.

As an illustration, suppose the first row of 10 observations in Table 1 can be associated with replenishment orders where a significant number of missions were canceled due to weather, creating reduced demand. This reduced demand is assumed to occur on 20% of replenishment orders; thus, demand can be considered to have two states: regular (with 80% probability) and low (20% of the time).

To demonstrate another approach to finding MTE approximations, the dataset in Table 1 will be used in this example to first determine a standard PDF that best fits the empirical data for each demand state. In this case, the log-normal distribution with $\mu = 1.03$

and $\sigma^2 = 0.31$ is selected for the regular state, and the LN(0.27,0.19) is chosen for state 2. The demand in each state for a given LT period is then a sum of i.i.d. log-normal random variables. This sum has no known distribution, but approximations for the PDF of a sum of log-normal random variables exist. Following Cobb et al. (2013), the Fenton-Wilkinson approximation (Fenton, 1960) is implemented, and MTE distributions are fit to these approximations for each state and each possible LT value. For state 1 and state 2, these functions are denoted by $\hat{f}^{(1)}_{X|\{L=l\}}$ and $\hat{f}^{(2)}_{X|\{L=l\}}$, respectively. The conditional PDF for LTD given $L = l$ is then calculated as

$$\hat{f}_{X|\{L=l\}}(x) = 0.8 \cdot \hat{f}^{(1)}_{X|\{L=l\}}(x) + 0.2 \cdot \hat{f}^{(2)}_{X|\{L=l\}}(x). \quad (10)$$

The PDF for LTD is constructed as in Equation 5. The new LTD distribution is bi-modal, as shown in Figure 5.

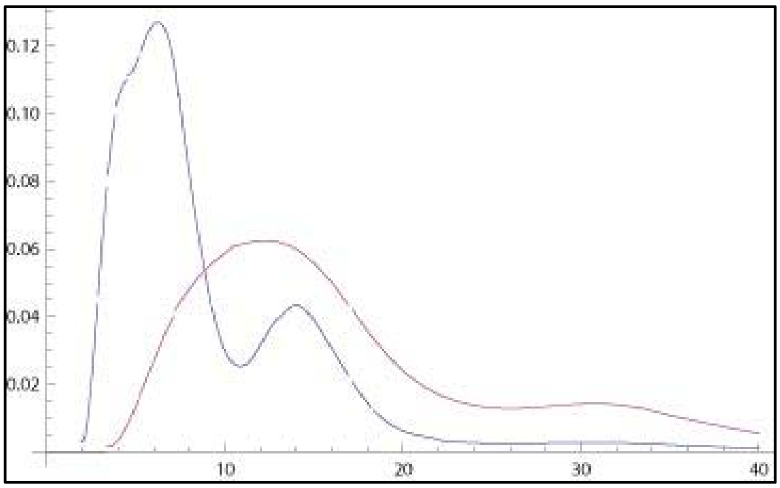

Figure 5. Mixture Distribution for Lead Time Demand With State-Dependent Demand

Suppose the state-dependent, bi-modal distribution shown in Figure 5 is the correct PDF for LTD. Using this distribution as part of the total cost function to find the optimal (*Q*,*R*) policy results in a 21% savings when compared to implementing the policies found earlier using the MTE distribution shown in Figure 4 (or one of the other approximations). The mixture distribution approach still yields a closed-form function for S_R and the optimization is still fast.

Coordinated Supply Chains

In this section, we consider a two-echelon supply chain, as depicted in Figure 6. A buyer experiencing random demand places its orders for inventory with the supplier.

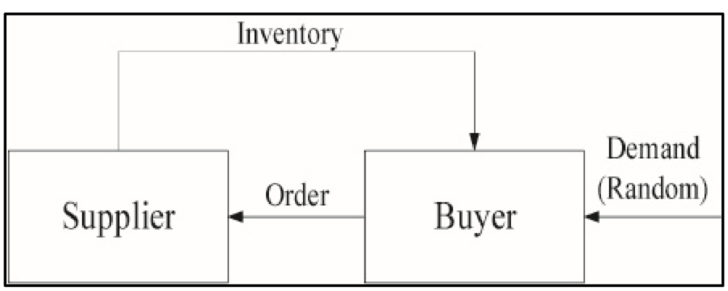

Figure 6. Two-Echelon Supply Chain
(Chaharsooghi & Heydari, 2010)

The cost function for the buyer in this problem is as follows:

$$TC_b(Q,R,V) = (K_b - V)\cdot\frac{Y}{Q} + \frac{\pi\cdot Y\cdot S_R}{Q} + h_s\cdot(0.5Q + R - E(X)). \quad (11)$$

Most of the notation is the same as for the cost function defined in Equation 7. The subscript *b* has been added to the fixed cost per order, annual unit holding cost, and total cost to identify this amount with the buyer. The subscript *s* will similarly represent the seller. The quantity *V* is a rebate provided by the seller to the buyer on a per order basis as an incentive for the buyer to adopt policies that benefit both parties (Cobb & Johnson, 2013). As discussed in the introduction, credit options and price discounts have also been considered in this two-level supply chain as coordination incentives (Chaharsooghi & Heydari, 2010; Chaharsooghi et al., 2011; Li & Liu, 2006).

The cost function for the supplier in this problem is

$$TC_s(Q,N,V) = \left(\frac{K_s}{N} + V\right)\cdot\frac{Y}{Q} + h_s(N-1)0.5Q. \quad (12)$$

In this two-level supply chain model, the buyer selects an order quantity and reorder point. The supplier receives orders of size *Q* from the buyer and purchases inventory from its vendors in a quantity that is an integer multiple *N* of the buyer's order size.

The supply chain can operate in one of three modes. First, the buyer can select Q_d and R_d without considering the effect of its selection on the supplier's costs. In response, the supplier selects N_d to minimize its own costs. This is referred to as the *decentralized* mode, and because there is no coordination, the rebate amount is $V = 0$. Total costs in the supply chain are $TC^d = TC_b(Q_d,R_d,0) + TC_s(Q_d,N_d,0)$. Second, the buyer and supplier can agree on values for Q_c, R_c, and N_c that minimize the sum of the cost functions in Equations 11 and 12. Because the members cooperate fully and are *centralized*, there is again no requirement for the supplier to provide a coordination incentive and $V = 0$. Total costs in this mode are denoted by $TC^c = TC_b(Q_c,R_c,0) + TC_s(Q_c,N_c,0)$.

If the parties are not centralized but can coordinate their policies, the potential exists to divide cost savings of $TC^+ = TC^d - TC^c$. An interval $[V_{min},V_{max}]$ can be calculated (Cobb & Johnson, 2013) such that any value for the rebate *V* in the interval reduces the total costs in the supply chain to centralized levels. The smallest value of the rebate the buyer will accept can be found by solving $TC_b(Q_c,R_c,V) = TC_b(Q_d,R_d,0)$ for *V*. This value is denoted by V_{min}. The largest value of the rebate the seller will accept can be found by solving $TC_s(Q_c,N_c,V) = TC_s(Q_d,N_d,0)$ for *V*. This value is denoted by V_{max}. For the example in this paper, we assume that if the parties agree to coordinate their policies (and implement Q_c, R_c, and N_c), the value of the rebate they select is $\bar{V} = (V_{min}+V_{max})/2$.

All of the two-echelon supply chain models referenced previously assume that demand for the entire LT period is normally distributed. For the case where both *Q* and *R* are selected to minimize total costs, Charharsooghi and Heydari (2010) derived expressions that state the optimal value for *Q* (in either the decentralized or centralized mode) as a function of the optimal value for *R* (and vice versa) and the standard normal cumulative density function. The optimal values can be found by iterating between these two expressions. The supplier selects the integer value for *N* that minimizes its costs subject to the choices of the buyer.

By implementing the mixture distribution approach, we can develop closed-form expressions for the cost functions in Equations 11 and 12 and find optimal solutions in the same manner as the solutions presented earlier in the paper for the (*Q*,*R*) inventory model. For illustration, assume $Y = E(D) \cdot 250$ working days = 720, $K_s = K_b = 30$, $h_s = h_b = 4$, and $\pi =$

5. These parameters are the same as used in the earlier example and the supplier has the same cost structure as the buyer (obviously, this may not always be true in practice).

For the previous example, employing the MTE mixture distribution in Figure 4 gives the same results in Table 2 for the decentralized case—Q_d = 110 and R_d = 27. In this mode, the supplier selects the multiple of the buyer's order quantity that minimizes its costs. Because $TC_s(110,1,0) = 197$ and $TC_s(110,2,0) = 316$, the supplier selects $N_d = 1$. Total supply chain costs in the decentralized mode are $TC^d = 678$.

In the centralized mode, we find the optimal order quantity and reorder point that minimizes $TC_b(Q,R,0)+TC_s(Q,N,0)$ for several possible values of N, then choose the optimal values that give the lowest combined supply chain cost. Again, using the MTE mixture distribution allows the construction of a closed-form total cost function, and optimization over this function in Mathematica is fast. Using the MTE mixture distribution, we find that Q_c=154, R_c=24, and N_c=1. Total supply chain costs in the centralized mode are TC^d= 648. Table 3 summarizes the optimal values for the decision variables in each mode and the total costs for each party and the supply chain. The answers obtained with the mixture distribution approach are compared with those obtained by using the solutions shown by Chaharsooghi and Heydari (2010).

Table 3. Optimal Solutions and Total Costs for the Supply Chain in Three Modes of Operation

Normal	Q	R	N	V	TC_b	TC_s	TC
Decentralized	108	25	1	0	483	200	683
Centralized	151	23	1	0	506	143	649
Coordinated	151	23	1	8.53	466	183	649

MTE Mixture	Q	R	N	V	TC_b	TC_s	TC
Decentralized	110	27	1	0	481	197	678
Centralized	154	24	1	0	507	141	648
Coordinated	154	24	1	8.51	467	181	648

A comparison of the solutions in the decentralized and centralized models shows that the costs in the entire supply chain can be reduced by $TC^+ = TC^d - TC^c = 30$ if the centralized order quantity and reorder point are implemented. However, these policies increase costs for the buyer by 507 - 481 = 26. By using the solutions in Cobb and Johnson (2013) to find the value $\bar{V}$ that divides the cost savings of operating in the centralized mode between the buyer and the seller, the buyer is adequately compensated for increasing its order quantity. The rebate amount for this problem is 8.51 per order cycle. Both members experience costs that are lower than in the decentralized mode.

Conclusions

This paper serves as an introduction to using a mixture distribution approach to modeling the probability density function for LTD in problems where a continuous review inventory system is implemented. First, construction of the lead time distribution was illustrated. This distribution was then utilized to determine optimal order policies in cases where a buyer makes its decisions alone, and then when members of a two-level supply chain coordinate their actions.

This paper represents the first stage of the research to be conducted for the project entitled “Modeling Uncertainty in Military Supply Chain Management Decisions,” which has been funded under BAA Number NPS-BAA-12-002 through the Naval Postgraduate School (Grant N00244-13-1-0014). For the expanded project, inventory requisition data for a five-year period has been obtained from the Air Force Standard Base Supply System for a power supply unit used on F-15 and F-16 aircraft. The techniques presented in this paper will be compared to an approach currently used by the Air Force that employs a negative binomial approximation to the lead time demand distribution. The comparison will be similar, but the hypothetical data in this paper will be replaced by the actual historical data provided by the Air Force.

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