A theoretical analysis of unsteady transonic cascade flow.

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Monterey, California. Naval Postgraduate School

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A THEORETICAL ANALYSIS OF UNSTEADY TRANSONIC CASCADE FLOW

Philip Robert Elder
THE THESIS

A THEORETICAL ANALYSIS OF UNSTEADY
TRANSONIC CASCADE FLOW

by

Philip Robert Elder

Thesis Advisor: M.F. Platzer

December 1972

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A Theoretical Analysis of Unsteady Transonic Cascade Flow

by

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Lieutenant Commander, United States Navy
B.S., United States Naval Academy, 1965

Submitted in partial fulfillment of the requirements for the degree of

AERONAUTICAL ENGINEER

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ABSTRACT

This thesis presents an analysis of transonic potential flow through an oscillating unstaggered thin plate cascade. A collocation technique is used involving the superposition of adjacent blade "isolated foil" potentials with interference potentials of unknown strength. Imposition of flow tangency requirements leads to integral equations for the unknown source distributions of the interference potentials. Results presented include the interference coefficients for the steady and oscillating cases.

The steady case is extended to the determination of the potentials along the blade and compares favorably with a parallel solution using a Laplace transform approach to the sonic wind tunnel wall interference problem.
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<td>L</td>
<td>the number of points on the blade at which the boundary conditions are enforced</td>
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\[ L(A) = \text{Laplace transform of } A \]
\[ N = \text{summation index} \]
\[ M = \text{Mach number} \]
\[ P = \text{blade spacing} \]
\[ p = \text{pressure} \]
\[ Q = \lambda + 2iK \]
\[ R = \text{gas constant} \]
\[ R = \text{nondimensional blade spacing} \]
\[ S = \text{Fresnel sine integral} \]
\[ s = \text{entropy of the individual particle} \]
\[ \text{SIN} = \text{sine} \]
\[ T = \text{absolute temperature} \]
\[ t = \text{time} \]
\[ U = \text{X velocity component} \]
\[ u = \text{perturbation X velocity} \]
\[ V = \text{Y velocity component} \]
\[ v = \text{perturbation Y velocity} \]
\[ W = \text{absolute velocity} \]
\[ X = \text{abscissa coordinate} \]
\[ X = \text{nondimensional abscissa coordinate} \]
\[ Y = \text{ordinate coordinate} \]
\[ Y = \text{nondimensional ordinate coordinate} \]
\[ \alpha = \text{maximum oscillation angle measured from the abscissa} \]
\[ \gamma = \text{ratio of specific heats} \]
\[ V = \text{interference function coefficient} \]
\[ \varepsilon = \text{a small quantity of perturbation magnitude} \]
\( \eta \) = ordinate coordinate
\( \theta_m \) = normal velocity function coefficient
\( \lambda \) = thickness parameter, \( \lambda = (\gamma+1)\phi_{1XX} \)
\( \xi \) = abscissa coordinate
\( \rho \) = density
\( \phi, \psi \) = velocity potential
\( \phi \) = perturbation potential
\( \phi_1 \) = perturbation potential, steady component
\( \phi_m \) = isolated blade perturbation potential
\( \phi_m \) = interference perturbation potential
\( \psi \) = perturbation potential, unsteady component
\( \omega \) = circular frequency
\( \frac{dA}{dB} \) = derivative of A with respect to B
\( \frac{\partial A}{\partial B}, A_B \) = partial of A with respect to B
\( \frac{DA}{DB} \) = substantial derivative of A with respect to B

**SUBSCRIPTS**

\( \omega \) = quantity evaluated upstream in steady unperturbed conditions
\( x,y,t \) = partial with respect to
\( u \) = upper
\( L \) = lower
\( l \) = steady state quantity
o = reference blade  III, A, B
l = adjacent blade  III

SUPERSCRIPT
*
* = critical condition, M = 1  II
+ = approaching from the positive side  II
- = approaching from the negative side  II

MISCELLANEOUS
\overline{\text{A}} = overbar, nondimensional quantity  II
\overline{\text{A}} = overbar, Laplace transform  III
\overline{\text{A}} = overbar, matrix  A
\hat{\text{A}} = hat, unsteady component  II

COMPUTER VARIABLES
COS = cosine
DK = reduced frequency, K
DL = \lambda
DLAM = \lambda
EPS = convergence criterion
ERFC(A) = 1 - erf(A)
EXP(A) = exponential function, e^A
PI = \pi
RHO = R
SIN = sine
XA = lower limit of an integral
XO = upper limit of an integral
ACKNOWLEDGEMENT

It is only through the guidance and infinite patience of Dr. M.F. Platzer that this contribution was made possible. It has been both a privilege and an honor to have participated in this small part of his life work.
I. INTRODUCTION

Considerable advances are still being made in the development of aircraft turbopropulsion systems. In the quest for higher thrust to weight ratios, one must look to higher stage loading as well as significantly increased turbine inlet temperatures. From the compressor and fan design point of view, this translates primarily into increased tip speeds. Moving toward better blade design, the investigation of transonic and supersonic relative flows is becoming more important not only for increased performance but also for flutter prediction. As shown by Pratt and Whitney, various flutter problems have already been encountered [Fig. 1 as extracted from Ref. 1] and more such problems can be expected in the future.

Recognizing that the actual three dimensional flow fields encountered in a transonic stage are complicated by the effects of rotation, finite blade thicknesses and cambers, boundary layers, and spanwise Mach distributions, it is excessively difficult to model these effects simultaneously. One can hope that eventually a complete analysis will become practical, but at this time the problems must be faced piecemeal. One simplified technique historically used in modeling such flows is the two dimensional cascade.

The following investigation was concerned with transonic cascade flow. This problem had previously been considered
by Hamamoto [Ref. 2] and Gorelov [Ref. 3]. Also of importance is the related problem of transonic wind tunnel interference. Here the studies of Sandeman [Ref. 4] and Murasaki [Ref. 5] must be mentioned. For a more general exposition of this problem, reference is made to Berndt [Ref. 6] and Goethert [Ref. 7].

Hamamoto based his analysis on the linearized oscillatory transonic flow equations. Gorelov attempted to include the effect of blade thickness by approximating the nonlinear transonic perturbation equation. This approximation, however, is valid only for low supersonic flows. In the present investigation, a transonic flow approximation was introduced based on previous work by Oswatitch [Ref. 8] and Teipel [Ref. 9]. The technique permitted a general formulation for both the steady choke flow problem (in conjunction with Spreiter's local linearization technique [Ref. 10]) and the oscillatory problem. The resulting boundary value problem was solved in a manner similar to that used by Gorelov. The problem formulation, solution technique, numerical evaluation of the recurring integrals, and representative results are described in the following sections.
II. UNSTEADY TRANSONIC FLOW THEORY

A. DEVELOPMENT OF THE GOVERNING TRANSONIC FLOW EQUATIONS

Consider planar flow of an inviscid adiabatic perfect gas along the positive X axis which is perturbed by a thin body (or thin bodies) executing small transverse unsteady motions. The perfect gas equation is applicable

\[ p = \rho RT \]  

(II-1)

where:

- \( p \) = pressure
- \( \rho \) = density
- \( T \) = absolute temperature
- \( R \) = gas constant
Furthermore the conservation conditions for mass, momentum, and energy must hold. Denoting the velocity components in the X and Y direction respectively as $U$ and $V$, the conservation of mass may be expressed as

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (U \rho)}{\partial x} + \frac{\partial (V \rho)}{\partial y} = 0 \quad \text{(II-2)}
\]

Conservation of momentum is expressed as

\[
\frac{DU}{Dt} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} \quad \text{(II-3)}
\]

\[
\frac{DV}{Dt} = \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial y}
\]

Conservation of energy is

\[
\frac{D}{Dt}[e + \frac{W^2}{2}] = - \frac{1}{\rho} \left[ \frac{\partial}{\partial x}(UP) + \frac{\partial}{\partial y}(VP) \right] \quad \text{(II-4)}
\]

or

\[
\frac{D s}{Dt} = 0 \quad \text{(II-5)}
\]

where

\[
\frac{D}{Dt} = \text{substantial derivative}
\]

$e$ = energy for the individual particle

$s$ = entropy for the individual particle

$W^2 = U^2 + V^2$
These equations, of course, exclude finite discontinuities (shocks) in the flow field.

As is well known, an important role is played by certain functions which reduce the number of dependent variables. Most often the velocity potential is introduced. Its existence depends on the condition of irrotationality of the flow. For transonic flows over sufficiently thin bodies (small perturbation assumption!) it is usually possible to work with the velocity potential since any vorticity introduced by occurring shocks is of at least third order in the perturbation velocities. It follows that the velocity can be expressed as

\[ U = \frac{\partial \phi}{\partial x} = \phi_x \quad \text{(II-6)} \]

\[ V = \frac{\partial \phi}{\partial y} = \phi_y . \]

The continuity equation (II-2) may be written

\[- \frac{1}{\rho} \frac{D\rho}{Dt} = \phi_{xx} + \phi_{yy}. \quad \text{(II-7)}\]

Recall that the velocity of sound is

\[ a^2 = \left( \frac{dP}{d\rho} \right)_{s=\text{constant}} \quad \text{(II-8)} \]
and that

\[
\frac{D\rho}{Dt} = \frac{d\rho}{dP}, \quad \frac{DP}{Dt} = \frac{1}{a^2} \frac{DP}{Dt}.
\]  (II-9)

Therefore the continuity equation (II-7) becomes

\[
\phi_{xx} + \phi_{yy} + \frac{1}{a^2\rho} (\phi_{x} P_x + \phi_{y} P_y + P_t) = 0. \quad (II-10)
\]

Integration of the Euler equations along a streamline gives the unsteady Bernoulli equation

\[
\phi_t + \frac{W^2}{2} + \int \frac{dP}{\rho} = F(t) \quad (II-11)
\]

where \( F(t) = \frac{1}{2} U_{\infty}^2 \) for uniform parallel flow at infinity.

Taking the partial of (II-11) with respect to \( t \), one has

\[
\frac{1}{\rho} P_t = \frac{3}{t} [\phi_t + \frac{W^2}{2}] . \quad (II-12)
\]

Inserting (II-3) and (II-12) for \( P_x, P_y, \) and \( P_t \) in (II-10) and utilizing (II-6) one obtains the following partial differential equation in \( \phi \).

\[
(1 - \frac{x^2}{a^2}) \phi_{xx} + (1 - \frac{y^2}{a^2}) \phi_{yy} - \frac{2\phi_{x} \phi_{y}}{a^2} \phi_{xy} - \frac{2\phi_{x}}{a^2} \phi_{xt} - \frac{2\phi_{y}}{a^2} \phi_{yt} - \frac{1}{a^2} \phi_{tt} = 0 \quad (II-13)
\]
The usual way to simplify this equation is to neglect all product terms obtaining

\[(1-M_\infty^2)\phi_{xx} + \phi_{yy} - \frac{2M_\infty}{a_\infty} \phi_{xt} - \frac{1}{a_\infty^2} \phi_{tt} = 0\] (II-14)

which is the well known unsteady linearized potential equation valid for the supersonic and subsonic regimes. However, for the case near \(M_\infty = 1\), such a "naive" linearization cannot, in general, be carried out. Consider the following analysis for the steady case.

Define the perturbation potential, \(\phi\), such that

\[\phi = U_\infty x + \phi.\] (II-15)

For the steady case, the Bernoulli equation (II-11) is

\[\frac{W^2}{2} + \frac{dP}{\rho} = \text{constant}.\] (II-16)

Now since

\[\frac{dP}{\rho} = C_p T = \frac{\gamma}{\gamma - 1} R T\]

and

\[a^2 = \left(\frac{dP}{\rho}\right)_s = \gamma R T\]

Equation (II-16) becomes
\[ \frac{W^2}{2} + \frac{a^2}{\gamma-1} = \text{constant} \quad (II-17) \]

The constant is evaluated by noting that at the critical Mach number

\[ W^* = a^* \]

and (II-17) becomes

\[ a^2 = \frac{\gamma+1}{2} a^*^2 - \frac{\gamma-1}{2} W^2 \quad (II-18) \]

With (II-18) and (II-15) in (II-13), one has

\[ \phi_{xx}(a^2_{\infty} - \nu^2) + \phi_{yy} a^2_{\infty} = \phi_{xx}[(\gamma+1)U_{\infty} \phi_x + \frac{\gamma+1}{2} \phi_x^2] + \frac{\gamma-1}{2} \phi_y^2 \]

\[ + 2\phi_{xy}[U_{\infty} \phi_y + \phi_x \phi_y] \]

\[ + \phi_{yy}[(\gamma-1)U_{\infty} \phi_x + \frac{\gamma-1}{2} \phi_x^2 + \frac{\gamma+1}{2} \phi_y^2] \quad (II-19) \]

As was mentioned previously, under the Prandtl assumption, one would neglect the product terms on the right-hand side (RHS). However, for \( M_{\infty} \approx 1 \), a more careful analysis is in order. To this end one introduces the Prandtl-Glauert transformation:
\[ \phi = \epsilon \bar{\phi}(X, \bar{Y}) \quad (II-20) \]

where \( \bar{X} = X \)
\[ \bar{Y} = Y \sqrt{1-M^{-2}} \quad (II-21) \]
\[ M_{\infty} = \frac{U_{\infty}}{a_{\infty}} \quad (II-22) \]

Here, one requires \( \bar{\phi} \) to be of order unity, therefore \( \epsilon \)
assumes the order of magnitude of \( \phi \).

Equation (II-18) becomes

\[
\epsilon \bar{\phi}_{xx}(1-M_{\infty}^{-2})a_{\infty}^2 + \epsilon \bar{\phi}_{yy}(1-M_{\infty}^{-2})a_{\infty}^2 = \epsilon^2(\gamma+1)U_{\infty}\bar{\phi}_{xx}\bar{\phi}_{x}^2 \\
+ \epsilon^3 \frac{y+1}{2} \frac{\phi_{xx}\bar{\phi}_{x}^2}{x} + \epsilon^3 \frac{y-1}{2} (1-M_{\infty}^{-2}) \frac{\phi_{xx}\bar{\phi}_{y}^2}{y} \\
+ 2\epsilon^2 \frac{\phi_{xy}(1-M_{\infty}^{-2})U_{\infty}\bar{\phi}_{y}^2}{x} + 2\epsilon^3 \frac{\phi_{xy}(1-M_{\infty}^{-2}) \bar{\phi}_{x}\bar{\phi}_{y}^2}{y} \\
+ \epsilon^2 (1-M_{\infty}^{-2})(\gamma-1)U_{\infty}\bar{\phi}_{x}\bar{\phi}_{y}^2 + \epsilon^3 \frac{\phi_{xy}(1-M_{\infty}^{-2}) \bar{\phi}_{yy}\bar{\phi}_{y}^2}{y} \\
+ \epsilon^3 \frac{(\gamma-1)}{2} (1-M_{\infty}^{-2}) \bar{\phi}_{yy}\bar{\phi}_{x}^2 \quad (II-23) \]

Now the terms on the RHS may be neglected for suitably
small \( \epsilon \). To investigate the validity of such a measure,
one may compare the largest term on the RHS with a term
from the LHS. This term is the first term of order \( \epsilon^2 \), the
remaining terms being of order \( \epsilon^3 \) or \( \epsilon^2(1-M_{\infty}^{-2}) \). Consider

---

\[ \text{Equation (II-18) becomes} \]
\[ \epsilon \bar{\phi}_{xx}(1-M_{\infty}^{-2})a_{\infty}^2 + \epsilon \bar{\phi}_{yy}(1-M_{\infty}^{-2})a_{\infty}^2 = \epsilon^2(\gamma+1)U_{\infty}\bar{\phi}_{xx}\bar{\phi}_{x}^2 \\
+ \epsilon^3 \frac{y+1}{2} \frac{\phi_{xx}\bar{\phi}_{x}^2}{x} + \epsilon^3 \frac{y-1}{2} (1-M_{\infty}^{-2}) \frac{\phi_{xx}\bar{\phi}_{y}^2}{y} \\
+ 2\epsilon^2 \frac{\phi_{xy}(1-M_{\infty}^{-2})U_{\infty}\bar{\phi}_{y}^2}{x} + 2\epsilon^3 \frac{\phi_{xy}(1-M_{\infty}^{-2}) \bar{\phi}_{x}\bar{\phi}_{y}^2}{y} \\
+ \epsilon^2 (1-M_{\infty}^{-2})(\gamma-1)U_{\infty}\bar{\phi}_{x}\bar{\phi}_{y}^2 + \epsilon^3 \frac{\phi_{xy}(1-M_{\infty}^{-2}) \bar{\phi}_{yy}\bar{\phi}_{y}^2}{y} \\
+ \epsilon^3 \frac{(\gamma-1)}{2} (1-M_{\infty}^{-2}) \bar{\phi}_{yy}\bar{\phi}_{x}^2 \quad (II-23) \]

Now the terms on the RHS may be neglected for suitably
small \( \epsilon \). To investigate the validity of such a measure,
one may compare the largest term on the RHS with a term
from the LHS. This term is the first term of order \( \epsilon^2 \), the
remaining terms being of order \( \epsilon^3 \) or \( \epsilon^2(1-M_{\infty}^{-2}) \). Consider
the ratio

$$E = \frac{U_\infty c(\gamma+1) \frac{\partial}{\partial x}}{a_\infty^2 (1-M_\infty^2)}$$  \hspace{1cm} (II-24)$$

It is clear that the nonlinear term becomes of higher order for the subsonic and supersonic cases but must be retained for transonic flows. This consideration can be generalized to unsteady transonic flows thus leading to the transonic small perturbation equation as the governing equation:

$$[1-M_\infty^2 - (\gamma+1) \frac{M_\infty^2}{U_\infty} \phi_x] \phi_{xx} + \phi_{yy} - 2 \frac{M_\infty}{a_\infty} \phi_{xt} - \frac{1}{a_\infty^2} \phi_{tt} = 0$$

$$\hspace{1cm} (II-25)$$

A rigorous justification for this equation was first given by Lin, Reissner, and Tsien [Ref. 11] and by Landahl [Ref. 12].

A further simplification of this equation is possible if the transonic flow is sufficiently unsteady. Although this can be shown by mathematical order of magnitude considerations in these same references, it is instructive to consider the physical aspects of the transonic wave propagation problem. Figure 3 shows an acoustic source in transonic flow. It may be seen that the parts of the wave fronts moving upstream, referred to as receding waves, move away very slowly from the source. The downstream or advancing waves move with a velocity approximately equal to twice the speed of sound.
Assuming now a continuous spatial distribution of disturbance sources, it is readily understood from Fig. 4 that the receding waves will tend to cancel provided the wavelength (given by $2\pi|a-U|/\omega$, where $\omega$ is the circular frequency of the disturbance) is very short compared to body reference length, C. However, if the sources oscillate very slowly,
pressure waves of the same sign will have sufficient time to interact and possibly to build up to finite amplitudes. Obviously, these receding waves may create large phase lags between motion and pressure on the airfoil causing severe aeroelastic instability problems.

We can draw the conclusion, therefore, that sufficiently unsteady transonic flows can be analyzed by linearized theory, Eqn. (II-14), whereas only mildly unsteady transonic flows must be investigated using Eqn. (II-25). The condition for linearization is generally stated as [Ref. 12]

\[ K \gg |1 - M_{\text{LOCAL}}| \]

where \( |1 - M_{\text{LOCAL}}| \) is the largest deviation of the Mach number from unity in the flow and \( K \) is the reduced frequency \( (K = \omega C/U_\infty) \).

B. DEVELOPMENT OF THE BOUNDARY CONDITIONS

To completely describe the problem, proper boundary and initial conditions must be specified. The most important ones are:

1. The flow on the surface of the body must be tangent at any instant of time.

2. Sommerfeld's radiation condition must be satisfied at any instant of time, i.e., waves propagate away from sources of disturbance.

Further, certain edge and wake conditions must be satisfied, e.g., smooth flow at trailing edges must be established (Kutta condition).
The general flow tangency condition can be stated as follows. If the equation of the surface of a body which is moving in a time dependent manner is given by

\[ F(X,Y,t) = 0, \]  

then the flow tangency condition requires that the total derivative of the function \( F \) is zero, i.e.,

\[ \frac{DF}{Dt} = F_t + U F_x + V F_y = 0. \]  

Consider an airfoil that is placed in a Cartesian coordinate system in such a way that if the airfoil is restricted to small amplitude motions out of the \( Y = 0 \) plane, then its upper and lower surface can be described by

\[ F_u = Y - Y_u(X,t) = 0 \]  

\[ F_L = Y - Y_L(X,t) = 0 \]  

Therefore, from the boundary condition one has

\[ V = Y_{ut} + U Y_{ux} \quad \text{at} \quad Y = Y_u(X,t) \]  

\[ V = Y_{Lt} + U Y_{Lx} \quad \text{at} \quad Y = Y_L(X,t) \]  

One now introduces the small perturbation assumption,

\[ \phi = U_\infty X + \phi \]  

where the disturbance velocity components are

\[ u = \phi_x \]  

\[ v = \phi_y \]
such that

\[
U = U_\infty + u \quad \text{(II-31)}
\]
\[
V = v \quad \text{(II-32)}
\]

and \( u, v \ll U_\infty \).

In (II-29) the term, \( u Y_u \), can then be neglected compared to the much larger term, \( U_\infty Y_u \). The boundary conditions then simplify to

\[
v = Y_{u\text{}} + U_\infty Y_u \quad \text{for } Y = Y_u(X,t)
\]

\[
v = Y_{L\text{}} + U_\infty Y_L \quad \text{for } Y = Y_L(X,t).
\]

For small amplitude oscillations, (II-33) can be further simplified by expressing the flow tangency condition in the plane \( Y = 0 \). Taylor series expansion gives

\[
v(X,Y_u,t) = v(X,0^+,t) + Y_u \frac{\partial v(X,0^+,t)}{\partial Y} + \ldots
\]

and

\[
v(X,Y_L,t) = v(X,0^-,t) + Y_L \frac{\partial v(X,0^-,t)}{\partial Y} + \ldots
\]

Using the same argument as above, the product terms can be neglected so that the flow tangency condition can be stated as

\[
v = Y_{u\text{}} + U_\infty Y_u \quad \text{for } Y = 0^+
\]

\[
v = Y_{L\text{}} + U_\infty Y_L \quad \text{for } Y = 0^-
\]

For a thin foil performing a time dependent motion about the \( Y = 0 \) plane, the airfoil thickness terms can be disregarded by similar arguments. The above conditions become
\[ v = \frac{\partial h}{\partial t} + U_\infty \frac{\partial h}{\partial x} \quad \text{for} \quad Y = 0 \quad (II-35) \]

where \( h(X,t) \) describes the time dependent chord line of the blade.

C. THE DERIVATION OF THE UNSTEADY EQUATIONS

Define the following nondimensional terms.

\[
\begin{align*}
\bar{X} &= \frac{X}{C} \\
\bar{Y} &= \frac{Y}{C} \\
\bar{t} &= \frac{t U_\infty}{C} \\
\bar{\phi} &= \frac{\phi}{U_\infty C} \\
\bar{\nu} &= \frac{\nu}{U_\infty} \\
\bar{h} &= \frac{h}{C}
\end{align*}
\]

where \( C \) is the chord of the foil. Equations (II-25) and (II-35) may now be written in nondimensional form:

\[
[1 - M_\infty^2 - (\gamma+1)M_\infty^2 \bar{\phi}_x] \bar{\phi}_{xx} + \bar{\phi}_{yy} - 2M_\infty \bar{\phi}_{xt} - M_\infty^2 \bar{\phi}_{tt} = 0
\]

\( (II-37) \)

and

\[ \bar{\nu} = \bar{h}_t + \bar{h}_x \]

\( (II-38) \)

Since the main interest of this thesis centers on the determination of the aeroelastic stability of a foil, only small perturbations from the steady state need be studied.
If one restricts oneself to such small amplitude perturbations, it is possible to write the perturbation potential as the sum of a steady state and a time dependent potential. Also, in the case of a harmonic oscillation, one may describe the time dependent quantities as the product of a complex amplitude function and a harmonic time factor. Therefore one defines the potentials \( \phi \), and \( \psi \) such that

\[
\bar{\phi}(\overline{x}, \overline{y}, t) = \phi_1(\overline{x}, \overline{y}) + \psi(\overline{x}, \overline{y}, K)e^{iKt} \quad \text{(II-39)}
\]

and the foil chord function and vertical velocity are

\[
\bar{h}(\overline{x}, t) = h_1(\overline{x}) + \hat{h}(\overline{x}, K)e^{iKt} \quad \text{(II-40)}
\]
\[
\bar{v}(\overline{x}, t) = v_1(\overline{x}) + \hat{v}(\overline{x}, K)e^{iKt} \quad \text{(II-41)}
\]

where

\[
K = \frac{\omega C}{U_{\infty}} \quad \text{(II-42)}
\]

It may be shown that (II-37) becomes

\[
[1-M_\infty^2 - M_\infty^2(\gamma+1)]\phi_{1x} + \frac{\phi_{1yy}}{\phi_{1xx}} = 0 \quad \text{(II-43)}
\]

which is the steady state equation, and the unsteady equation is

\[
[1-M_\infty^2]\psi_{xx} - M_\infty^2(\gamma+1)\frac{\partial}{\partial x} (\phi_{1x}\psi_x) + M_\infty^2K^2\psi - 2iKM_\infty^2\psi_x + \frac{\psi_{yy}}{\psi_{xx}} = 0 \quad \text{(II-44)}
\]

The boundary equation (II-38) becomes

\[
v_1(\overline{x}) = h_1(\overline{x}) \quad \text{(II-45)}
\]

and

\[
\hat{v}(\overline{x}, K) = \hat{h}_X(\overline{x}, K) + iK\hat{h}(\overline{x}, K) \quad \text{(II-46)}
\]
On the foil surface equation (II-43) must satisfy the boundary equation (II-45); (II-44) must satisfy (II-46).

For convenience, the overbar and hat will now be dropped and the steady and unsteady equations with boundary equations become

\[ [1-M_{\infty}^2 - M_{\infty}^2(\gamma+1)] \phi_{1x} \phi_{1xx} + \phi_{1yy} = 0 \]  \hspace{1cm} (II-47)

subject to

\[ v_1 = h_{1x} , \hspace{1cm} \text{at} \ Y=0 \]  \hspace{1cm} (II-48)

and

\[ [1-M_{\infty}^2] \psi_{xx} - M_{\infty}^2(\gamma+1) \frac{\partial}{\partial x} (\phi_{1x} \psi_x) + M_{\infty}^2 K^2 \psi \]

\[ - 2iKM_{\infty}^2 \psi_x + \psi_{yy} = 0 \]  \hspace{1cm} (II-49)

subject to

\[ v = h_x + iKh , \hspace{1cm} \text{at} \ Y=0 . \]  \hspace{1cm} (II-50)

Note that (II-47) is a nonlinear equation in \( \phi_1 \). Equation (II-49) is linear in \( \psi \) but the existence of the variable coefficient \( \phi_{1x} \) considerably complicates its solution.

D. A BRIEF REVIEW OF POSSIBLE APPROACHES TO UNSTEADY TRANSONIC FLOW ANALYSIS

The following is a presentation of various approaches to the solution of equation (II-49).

1. Method of Teipel and Hosokawa

An approximation of equation (II-49) was proposed by Teipel [Ref. 9] and Hosokawa [Ref. 13] for oscillating
airfoils in sonic flow \((M_\infty = 1)\). For \textit{steady} sonic flow, Oswatitch [Ref. 8] introduced a linearized approximation by setting

\[(\gamma+1)\phi_{1xx} = \lambda = \text{constant}, \quad \lambda > 0, \quad \text{(II-51)}\]

allowing (II-43) to be written in the easily solved form

\[\phi_{1yy} - \lambda \phi_{1x} = 0. \quad \text{(II-52)}\]

Teipel and Hosokawa generalized this procedure to unsteady sonic flow by approximating equation (II-49) with (neglecting the term in \(\psi_{xx}\))

\[-\lambda \psi_x + \psi_{yy} - 2iK\psi_x + K^2\psi = 0. \quad \text{(II-53)}\]

Defining \(Q\)

\[Q = \lambda + 2iK, \quad \text{(II-54)}\]

one has from (II-46)

\[\psi_{yy} - Q \psi_x + K^2 \psi = 0. \quad \text{(II-55)}\]

This equation allows the study of sonic flow past a single oscillating airfoil in unbounded flow over the entire frequency range. It contains the steady solution (Oswatitch approximation) as a special case. For large \(K\) (Refer to page 23) the term containing \(\phi_{1xx}\) becomes negligibly small; therefore, retention of \(\lambda\) at high reduced frequencies becomes unnecessary. Thus, when one takes \(\lambda = 0\), the Teipel-Hosokawa theory also contains the linearized theory.
2. Local Linearization Technique for Steady Transonic Flow

It was shown that the Oswatitch approximation of the steady transonic small perturbation equation is

$$\phi_{1yy} - \lambda \phi_{1x} = 0, \quad \lambda = \text{constant} > 0. \quad (II-52)$$

The solution to this equation for sonic flow past a single airfoil is

$$\phi_1(x,y) = \frac{1}{\sqrt{\pi \lambda}} \int_{-1}^{x} \frac{V(s)}{\sqrt{x-s}} e^{-\frac{\lambda y^2}{4(x-s)}} ds. \quad (II-56)$$

Of particular interest is the velocity on the profile which is given by

$$\phi_{1x}(x,0) = \frac{1}{\sqrt{\pi \lambda}} \frac{d}{dx} \int_{-1}^{x} \frac{V(s)}{\sqrt{x-s}} ds. \quad (II-57)$$

Recalling the definition of $\lambda$, (II-51), Spreiter argued that the original variable, $\phi_{1xx}$, could be reintroduced into equation (II-57) thus linearizing the equation locally. This leads to an ordinary nonlinear differential equation in $\phi_{1x}$ which can be solved analytically. The agreement with experiment is remarkably good [Ref. 10]. Its application to the choked wind tunnel wall interference problem was given by Sandeman [Ref. 4]. It is expected that an extension of this technique to the unsteady case may prove fruitful.

3. WKBJ - Method

Rubbert [Ref. 14] applied the WKBJ method to this problem. Differentiating the axisymmetric counterpart of
equation (II-49) for $M_\infty = 1$, multiplying by $\phi_{1x}$ and applying the Hankel transform, the equation can be reduced to the form

$$\psi_{xx} + N^2 q(X,N) \psi = 0$$  \hspace{1cm} (II-50)

where $N$ is a large parameter. Unfortunately no inversion of the resulting transform integrals has been found.

4. Method of Geometrical Acoustics

A third approach has recently been proposed to obtain the thickness effects for oscillating wings in transonic flow [Ref. 15]. This procedure begins with a given non-uniform steady transonic flow field which has been determined either analytically or experimentally (In the latter case an analytical expression must be obtained which matches the measured pressure coefficient.). Superimposing upon this flow field small unsteady perturbations, the path and transmission time of these acoustic perturbations are determined by the methods of geometrical acoustics. Again, only preliminary calculations have been carried out to date.

5. Unsteady Transonic Flow with Shock Waves

Typically, transonic flows around an airfoil exhibit locally supersonic regions which are terminated by shock waves. Often, the interactions of small unsteady perturbations with these shock waves may be responsible for aerodynamically induced instabilities. Eckhaus [Ref. 16] made the problem mathematically tractable by assuming a normal shock extending from the surface to infinity with uniform
steady supersonic flow in front and uniform steady subsonic flow behind the shock. Performing a small perturbation analysis, i.e., superimposing small time dependent perturbations upon this idealized steady transonic flow, he was able to obtain equations for the interaction of the sound waves with the shock. However, only a few numerical results were obtained.

For the following study, the Teipel-Hosokawa approach was used.
III. TRANSONIC UNSTEADY CASCADE FLOW PROBLEM

A. PROBLEM FORMULATION FOR SONIC FLOW PAST OSCILLATING UNSTAGGERED CASCADES

Consider an infinite cascade of thin blades arranged as shown in Figure 5. Blade chord is $2b$ and blade spacing is $P$. Subscript zero refers to the reference blade and subscript one refers to the adjacent blade in the positive ordinate ($\eta$) direction. The reference coordinate system is fixed at mid-chord with the abscissa ($\xi$) in the direction of flow.

Two $\eta$ coordinates are specified with $\eta_0$ originating on the reference blade and $\eta_1$ on the adjacent blade such that

$$\eta_1 = \eta_0 + P$$  \hspace{1cm} (III-1)

Now the following nondimensional variables are defined:

$$X = \frac{\xi}{b}$$  \hspace{1cm} (III-2)
\[ y_m = \frac{n}{b}, \ m=0,1 \quad (III-3) \]

\[ R = \frac{P}{b} \quad (III-4) \]

As the governing equation, the Teipel approximation is used,

\[ \psi_{yy} - Q \psi_x + K^2 \psi = 0 . \quad (II-55) \]

This equation must be solved subject to the following boundary equations:

1. Zero disturbance upstream of the cascade.
2. Flow tangency on the blades.

As shown in Section II-B, this latter requirement can be expressed mathematically as follows. Along the reference blade equation (II-50) may be written

\[ \psi_y(X,Y_o=0) = v_o = \frac{\partial h_o}{\partial X} + iK h_o \quad (III-5) \]

and along the adjacent blade

\[ \psi_y(X,Y_o=R) = v_1 = \frac{\partial h_1}{\partial X} + iK h_1 . \quad (III-6) \]

To solve this boundary value problem, the potential function \( \psi \) is written as the summation

\[ \psi(X,Y_o) = \phi_o^o(X,Y_o) + \phi_0(X,Y_o) + \phi_1^o(X,Y_1) \]

\[ + \phi_1(X,Y_1) \quad (III-7) \]

where \( \phi_o^o \) and \( \phi_1^o \) are the isolated blade solutions for the reference and adjacent blades and \( \phi_o \) and \( \phi_1 \) are interference potentials.
Such an approach has previously been proposed for the subsonic and supersonic cases by D. N. Gorelov [Refs. 3 and 17]. Also refer to Samoylovich [Ref. 18].

Inserting (III-7) into (III-5) and (III-6) and realizing

\[ \phi_{my}(X,0) = v_m, \quad m=0,1, \]

one obtains for the reference blade,

\[ \phi_{o_0}(X,Y_0 = 0) + \phi_{1_0}(X,Y_1 = -R) + \phi_{1_0}(X,Y_1 = -R) = 0 \]

(III-8)

and for the adjacent blade,

\[ \phi_{o_0}(X,Y_0 = R) + \phi_{o_0}(X,Y_0 = R) + \phi_{1_0}(X,Y_1 = 0) = 0. \]

(III-9)

The isolated blade solution is already known from Teipel [Ref. 9] as

\[ \phi_0(X,Y_0) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} v_0(S) \frac{\exp\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4} \frac{Y_0^2}{X-S}\right]}{\sqrt{X-S}} \, dS \]

(III-10)

and

\[ \phi_1(X,Y_1) = \frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} v_1(S) \frac{\exp\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4} \frac{Y_1^2}{X-S}\right]}{\sqrt{X-S}} \, dS \]

(III-11)

The interference potentials must have the same form but contain unknown source distributions \( u_0 \) and \( u_1 \); thus,

\[ \phi_0(X,Y_0) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} u_0(S) \frac{\exp\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4} \frac{Y_0^2}{X-S}\right]}{\sqrt{X-S}} \, dS \]

(III-12)
and
\[
\phi_1(x, y_1) = \frac{1}{\sqrt{\pi Q}} \int_{-1}^{x} u_1(s) \frac{\exp\left[\frac{K^2}{Q(x-s)} - \frac{Q}{4} \frac{y_1^2}{x-s}\right]}{\sqrt{x-s}} ds. \quad (III-13)
\]

These source distributions are approximated by the power series as follows:

\[
u_m = \sum_{N=0}^{L-1} p_{mN} x^N
\quad (III-14)
\]

and

\[
u_m = \sum_{N=0}^{L-2} \theta_{mN} x^N
\quad (III-15)
\]

Inserting these approximations, (III-14) and (III-15), into equations (III-8) and (III-9) results in

\[
\sum_{N=0}^{L-1} \left\{ p_{oN} x^N + \frac{R}{2} \sqrt{\frac{Q}{\pi}} (\theta_{1N} + p_{1N}) \int_{-1}^{x} s^N (x-s)^{3/2} \exp\left[\frac{K^2}{Q(x-s)} - \frac{Q}{4} \frac{R^2}{x-s}\right] ds \right\} = 0
\quad (III-16)
\]

for the reference blade and

\[
\sum_{N=0}^{L-1} \left\{ p_{1N} x^N + \frac{R}{2} \sqrt{\frac{Q}{\pi}} (\theta_{0N} + p_{0N}) \int_{-1}^{x} s^N (x-s)^{3/2} \exp\left[\frac{K^2}{Q(x-s)} - \frac{Q}{4} \frac{R^2}{x-s}\right] ds \right\} = 0
\quad (III-17)
\]

for the adjacent blade.
The coefficients in (III-15) may be evaluated for the case of simple oscillation, no translation, and $\pi$ phase lag between blades as follows. One arbitrarily sets the angle of maximum amplitude for the reference blade at $\alpha_o = -1$ [Refer to Fig. 6]. Then the angle of the adjacent blade is $\alpha_1 = 1$. For small $\alpha$, equation (III-5) may be written

$$\psi_y(X,Y_o = 0) = -1 - iKX$$

(III-18)

and (III-6),

$$\psi_y(X,Y_1 = 0) = 1 + iKX$$

(III-19)
Hence, the coefficients are

\[ \theta_{00} = -1 \]
\[ \theta_{01} = -iK \]
\[ \theta_{10} = 1 \]
\[ \theta_{11} = iK . \]  

(III-20)

and

Similarly, for in phase oscillation, the coefficients are

\[ \theta_{00} = \theta_{10} = -1 \]

and

\[ \theta_{01} = \theta_{11} = -iK . \]

(III-21)

Satisfying boundary equations (III-16) and (III-17) at L points along the blades generates a set of 2L equations in 2L unknown interference coefficients. The coefficients in hand, \( \psi(X,Y_0) \) in (III-7) becomes a known function.

A computer program for the solution of this problem in the \( \pi \) phase shift case is developed in Appendix A. Some sample results are presented in Table II for the steady case and Table VII for the unsteady case.

B. EVALUATION OF THE INTEGRALS OCCURRING IN EQUATIONS (III-16) and (III-17)

The integrals occurring in (III-16) and (III-17) are of the form

\[ I_N = \int_{-1}^{X} \frac{S^N}{(X-S)^{3/2}} \exp\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4} \frac{R^2}{(X-S)}\right] dS . \]  

(III-22)

By making the substitution, \( U = X-S \), one obtains

\[ I_N = \int_{0}^{X+1} \frac{(X-U)^N}{U^{3/2}} \exp\left[\frac{K^2}{Q} U - \frac{Q}{4} \frac{R^2}{U}\right] dU \]  

(III-23)
Recalling $Q = \lambda + 2iK$, $I_N$ is complex and must be resolved into real and imaginary parts for evaluation.

The exponential term in (III-23) may be written

$$\exp\left[\frac{k^2}{Q} U - \frac{Q}{4} \frac{R^2}{U}\right] = \exp\left[\frac{-\lambda K^2 U}{\lambda^2 + 4K^2} - \frac{\lambda R^2}{4U}\right]$$

$$\cdot \exp\left\{i\left[-\frac{2K^3 U}{\lambda^2 + 4K^2} - \frac{KR^2}{2U}\right]\right\} \quad (III-24)$$

One now defines the following terms:

$$A = \frac{k^2}{\lambda^2 + 4K^2}$$

$$B = \frac{\lambda R^2}{4}$$

$$E = \frac{KR^2}{2}$$

Equation (III-24) may be rewritten,

$$\exp\left[\frac{k^2}{Q} U - \frac{Q}{4} \frac{R^2}{U}\right] = \exp[\lambda AU - B]\exp\{i[-2KAU - E]\} \quad (III-25)$$

Separating (III-25) into its real and imaginary parts, one obtains the following two fundamental integrals from (III-23):

$$I_{NR} = \int_0^{x+1} \frac{(X-U)^N}{U^{3/2}} \exp[\lambda AU - B]\cos[2KAU + E] \, dU \quad (III-26)$$

and

$$I_{NI} = \int_0^{x+1} \frac{(X-U)^N}{U^{3/2}} \exp[\lambda AU - B]\sin[2KAU + E] \, dU \quad (III-27)$$

where

$$I_N = I_{NR} - iI_{NI} \quad (III-28)$$
Expanding \((X-U)^N\) in the finite binomial series,

\[
(X-U)^N = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} N! x^{N-J+1} U^{J-1}}{(J-1)! (N-J+1)!}
\]  

equations (III-26) and (III-27) become

\[
I_{NR} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} N! x^{N-J+1} U^{J-1}}{(J-1)! (N-J+1)!} \int_0^{X+1} \left\{ U^{J-\frac{5}{2}} \exp[\lambda U - \frac{B}{U}] \cdot \cos[2\lambda U + \frac{E_1}{U}] \right\} dU
\]

and

\[
I_{NI} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} N! x^{N-J+1} U^{J-1}}{(J-1)! (N-J+1)!} \int_0^{X+1} \left\{ U^{J-\frac{5}{2}} \exp[\lambda U - \frac{B}{U}] \cdot \sin[2\lambda U + \frac{E_1}{U}] \right\} dU
\]

The fundamental integrals in (III-30) and (III-31) do not lend themselves to analytical integration. Numerical integration techniques may be used but care must be exercised in the vicinity of the lower limit. Here the integrand is singular when \(J < 5/2\). The problem is further aggravated by the rapidly oscillatory character of the functions \(\sin[\lambda U - \frac{B}{U}]\) and \(\cos[\lambda U - \frac{B}{U}]\).

Three specific cases are posed for the treatment of these integrals.

1. The Steady Case

Setting \(K = 0\) reduces (III-30) and (III-31) to the following:

40
\[ I_{NR} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} N! X^{N-J+1}}{(J-1)! (N-J+1)!} \int_0^{X+1} U^{J-\frac{5\epsilon}{2}} \exp[-\frac{B}{U}] \, dU \quad (III-32) \]

and

\[ I_{NI} = 0 \]

The numerical integration of this case is straightforward. The exponential term forces the integrand to zero near the origin permitting one to neglect that portion of the integral very near \( U = 0 \). Equation (III-32) may also be evaluated by expressing it in terms of the error function.

(Note: complete agreement between the two integration techniques has been obtained thus verifying the numerical integration procedure for the steady case.)

2. The Oscillatory Case with \( \lambda = 0 \)

In the unsteady case for \( \lambda = 0 \), equations (III-30) and (III-31) contain the basic integrals

\[ \int_0^{X+1} U^{J-\frac{5\epsilon}{2}} \cos[2K\mu + \frac{E}{U}] \, dU \quad (III-33) \]

and

\[ \int_0^{X+1} U^{J-\frac{5\epsilon}{2}} \sin[2K\mu + \frac{E}{U}] \, dU. \quad (III-34) \]

To avoid the previously mentioned difficulties near the lower limit, the integration must be accomplished in two steps, i.e.,
\[ \int_{0}^{X+1} \frac{u^{J-\frac{1}{2}} \cos(2K\alpha u + \frac{E}{U})}{\sin(2K\alpha u + \frac{E}{U})} du = \int_{0}^{a} \frac{u^{J-\frac{1}{2}} \cos(2K\alpha u + \frac{E}{U})}{\sin(2K\alpha u + \frac{E}{U})} du + \int_{a}^{X+1} \frac{u^{J-\frac{1}{2}} \cos(2K\alpha u + \frac{E}{U})}{\sin(2K\alpha u + \frac{E}{U})} du. \]

(III-35)

For judicious selection of \(a\), the second integral in the RHS may be numerically integrated in a straightforward manner. The integral on the limits 0 to \(a\), which will henceforth be referred to as the correction integral, may be approximated for sufficiently small \(a\) as follows.

Recall the following relations:

\[ \cos[2K\alpha u + \frac{E}{U}] = \cos[2K\alpha u] \cos[\frac{E}{U}] - \sin[2K\alpha u] \sin[\frac{E}{U}] \]  

(III-36)

and

\[ \sin[2K\alpha u + \frac{E}{U}] = \sin[2K\alpha u] \cos[\frac{E}{U}] + \cos[2K\alpha u] \sin[\frac{E}{U}] \]  

(III-37)

and the truncated series expansions

\[ \cos[2K\alpha u] = 1 - \frac{(2K\alpha u)^2}{2} \]  

(III-38)

and

\[ \sin[2K\alpha u] = 2K\alpha u - \frac{(2K\alpha u)^3}{6}. \]  

(III-39)

Equations (III-38) and (III-39) are valid for \(2K\alpha u\) sufficiently close to zero. One may now write
\[ \int_0^a u^{J-\frac{v}{2}} \cos[2kau + \frac{E}{U}]du = \]

\[ \int_0^a u^{J-\frac{v}{2}} \cos[\frac{E}{U}]du - (2ka) \int_0^a u^{J-\frac{v}{2}} \sin[\frac{E}{U}]du \]

\[ - \frac{(2ka)^2}{2} \int_0^a u^{J-\frac{v}{2}} \cos[\frac{E}{U}]du \]

\[ + \frac{(2ka)^3}{6} \int_0^a u^{J+\frac{v}{2}} \sin[\frac{E}{U}]du \]

(III-40)

and

\[ \int_0^a u^{J-\frac{v}{2}} \sin[2kau + \frac{E}{U}]du = \]

\[ \int_0^a u^{J-\frac{v}{2}} \sin[\frac{E}{U}]du + (2ka) \int_0^a u^{J-\frac{v}{2}} \cos[\frac{E}{U}]du \]

\[ - \frac{(2ka)^2}{2} \int_0^a u^{J-\frac{v}{2}} \sin[\frac{E}{U}]du \]

\[ - \frac{(2ka)^3}{6} \int_0^a u^{J+\frac{v}{2}} \cos[\frac{E}{U}]du . \]

(III-41)

In theory integrals of the form

\[ \int_0^a u^{m/2} \cos \frac{E}{U} \sin \frac{E}{U}du \]

may be expressed in terms of the Fresnel integrals. Hence, integration of (III-40) and (III-41) can be accomplished in "closed" form. (In actuality there is a computer accuracy limit on this technique as described in Appendix A-IV.)
Verification of this technique for \( J = 1 \) was accomplished by checking it against a known analytical function as follows:

\[
\int_{0}^{a} \frac{1}{U^{3/2}} \cos(2K\lambda + \frac{E}{U})dU + \int_{a}^{\infty} \frac{1}{U^{3/2}} \cos(2K\lambda + \frac{E}{U})dU = \\
\int_{0}^{\infty} \frac{1}{U^{3/2}} \cos(2K\lambda + \frac{E}{U})dU \tag{III-42}
\]

where the correction integral was evaluated using the Fresnel integral technique, the second integral was evaluated using numerical integration techniques, and the integral on the RHS was evaluated using the following relation:

\[
\int_{0}^{\infty} \frac{1}{U^{3/2}} \cos(\varphi U + \frac{b}{U})dU = \sqrt{\pi/b} \cos(\frac{\pi}{4} + \sqrt{2gb}) \] \tag{III-43}

For the case \( K = 0.01 \) and \( \lambda = 0.01 \), Table IA presents the comparison of the value of (III-43) with that obtained from (III-42) for various \( R \)'s. (The numerical integration was truncated at an upper limit of 500 and the convergence criterion EPS for subroutine ZSUM was set at 1.0E-4.)

Table IB displays the computed integral using (III-35) with an upper limit of \( X+1 = 2.0 \) for various values of \( (a) \). (The convergence limit placed on ZSUM was set at 1.0E-5.)

Table variables are listed below.

\[ DK = K \]
\[ DL = \lambda \]
\[ RHO = R \]
AL = a (Correction integral upper limit.)

SCOR = Value of correction integral (III-41) for J = 1.

CCOR = Value of correction integral (III-40) for J = 1.

SUMI = Value of numerical integration term from (a) to specified upper limit for the SIN integral.

SUMR = Value of numerical integration term from (a) to specified upper limit for the COS integral.

SINT = Value of (III-35) SIN integral for specified upper limit and J = 1.

CINT = Value of (III-35) COS integral for specified upper limit and J = 1.

SSIN = Value of (III-43) SIN integral.

CCOS = Value of (III-43) COS integral.

3. The General Case, $K \neq 0$ and $\lambda \neq 0$

In the apparent absence of a suitable correction integral for this case, one must evaluate the full equations (III-26) and (III-27) by numerical methods. The difficulty close to the lower limit is alleviated to some extent by the exponential term. The exponential function drives the amplitude of the integrand to zero more rapidly than a negative power of $U$ can amplify the highly oscillatory behavior of the transcendental functions. For this reason, as in the case of $K = 0$, one may justify neglecting that portion of the integrals very near zero.

Although no results are presented, the computer routine developed in Appendix A is designed to carry out such a case.
<table>
<thead>
<tr>
<th>R</th>
<th>SCOR</th>
<th>SUMI</th>
<th>SINT</th>
<th>SSIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>34.62</td>
<td>1.016</td>
<td>35.63</td>
<td>35.57</td>
</tr>
<tr>
<td>1.0</td>
<td>14.45</td>
<td>3.458</td>
<td>17.91</td>
<td>17.85</td>
</tr>
<tr>
<td>1.5</td>
<td>4.968</td>
<td>7.032</td>
<td>12.00</td>
<td>11.94</td>
</tr>
<tr>
<td>2.0</td>
<td>-1.114</td>
<td>10.16</td>
<td>9.045</td>
<td>8.987</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>R</th>
<th>CCOR</th>
<th>SUMR</th>
<th>CINT</th>
<th>CCOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
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<td>17.56</td>
<td>17.60</td>
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<tr>
<td>1.5</td>
<td>-5.797</td>
<td>17.45</td>
<td>11.65</td>
<td>11.69</td>
</tr>
<tr>
<td>2.0</td>
<td>-4.490</td>
<td>13.19</td>
<td>8.696</td>
<td>8.736</td>
</tr>
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**COMPARISON OF VALUES COMPUTED USING EQUATION (III-43) WITH THOSE OBTAINED USING (III-42) FOR AN UPPER LIMIT OF 500.**
<table>
<thead>
<tr>
<th>DK = 0.500</th>
<th>DL = 0.0</th>
<th>RHO = 0.500</th>
<th>AL = 0.010</th>
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<td>SCOR = 1.57005D</td>
<td>CCOR = 1.67114D-01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUMI = 3.96851D</td>
<td>SUMR = 3.24013D 00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SINT = 5.53856D</td>
<td>CINT = 3.40724D 00</td>
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<td></td>
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<table>
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<tr>
<th>DK = 0.500</th>
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<td></td>
</tr>
<tr>
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<td>SUMR = 6.06207D 00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SINT = 5.53856D</td>
<td>CINT = 3.40724D 00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>RHO = 0.500</th>
<th>AL = 0.100</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCOR = 3.74310D</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>SUMI = 1.79545D</td>
<td>SUMR = 4.53526D 00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SINT = 5.53856D</td>
<td>CINT = 3.40724D 00</td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
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<th>DL = 0.0</th>
<th>RHO = 0.500</th>
<th>AL = 0.200</th>
</tr>
</thead>
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<td>CCOR = 4.96270D-01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUMI = 9.18297D-01</td>
<td>SUMR = 2.91097D 00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SINT = 5.53856D</td>
<td>CINT = 3.40724D 00</td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>DK = 0.500</th>
<th>DL = 0.0</th>
<th>RHO = 0.500</th>
<th>AL = 0.300</th>
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<tr>
<td>SCOR = 4.87826D</td>
<td>CCOR = 1.27508D 00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUMI = 6.60294D-01</td>
<td>SUMR = 2.13216D 00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SINT = 5.53856D</td>
<td>CINT = 3.40724D 00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
C. CALCULATION OF THE POTENTIAL FUNCTION AND THE PRESSURE COEFFICIENT

The potential function is obtained from equation (III-7),

\[ \psi(X,Y_o) = \phi_0^0(X,Y_o) + \phi_0(X,Y_o) + \phi_1^0(X,Y_1) + \phi_1(X,Y_1) \]  

(III-7)

The interference potentials \( \phi_0 \) and \( \phi_1 \) may now be computed using the previously determined coefficients \( (P_{mN}, \ m = 0,1; \ N = 0,\ldots,L-1) \) in equations (III-12) and (III-13). Equation (III-7) may be rewritten in terms of the coefficients as follows:

\[ \psi(X,Y_o) = \sum_{N=0}^{L-1} \left\{ -\frac{1}{\sqrt{\pi}Q} \left( \theta_{0N} + P_{0N} \right) \int_{-1}^{1} \frac{X^N}{\sqrt{X-S}} \exp\left[ \frac{K^2}{Q}(X-S)^2 \right] \frac{Y_o^2}{(X-S)^2} dS \right\} \]

\[ + \frac{1}{\sqrt{\pi}Q} \left( \theta_{1N} + P_{1N} \right) \int_{-1}^{1} \frac{X^N}{\sqrt{X-S}} \exp\left[ \frac{K^2}{Q}(X-S)^2 \right] \frac{Y_1^2}{(X-S)^2} dS \]  

(III-44)

The resulting integrals may be evaluated using the same techniques utilized in the coefficient derivation.

The pressure coefficient is obtained from the relation

\[ C_p = -2(\psi_x + iK\psi) \]  

(III-45)

By placing the partial with respect to X of equation (III-44) into (III-45), one may obtain the pressure distribution at any point in the flow field as well as on the blade surfaces. Further, the force and moment coefficients are easily obtained by integration over the blade chord.
D. EVALUATION OF THE STEADY STATE POTENTIAL ALONG THE BLADE

The computation of the steady state potential along the upper surface of the reference blade follows from equation (III-44) evaluated at \( K = 0, Y_o = 0, \) and \( Y_1 = -R. \) The development which follows is for the \( \pi \) phase shift case where [From Eqns. (III-20).]

\[
\theta_{oo} = -1 \quad \text{(III-46)}
\]

\[
\theta_{lo} = 1
\]

Equation (III-44) becomes

\[
\psi_{K=0}(x,0) = \frac{1}{\sqrt{\pi \lambda}} \left\{ \int_{-1}^{1} \frac{1}{\sqrt{X-S}} \, dS + \int_{-1}^{1} \frac{\exp[-\frac{\lambda}{4} \frac{R^2}{X-S}]}{\sqrt{X-S}} \, dS \right\} \\
+ \frac{1}{\sqrt{\pi \lambda}} \sum_{N=0}^{L-1} \left\{ \psi_{1N} \int_{-1}^{1} \frac{X}{\sqrt{X-S}} \exp[-\frac{\lambda}{4} \frac{R^2}{X-S}] \, dS \right\} \\
- \psi_{0N} \int_{1}^{X} \frac{S^N}{\sqrt{X-S}} \, dS \right\}. \quad \text{(III-47)}
\]

The computer program is documented in Appendix B-I. The resulting potential values are compared with those obtained from the Sandeman wall interference solution of Section III-E in Tables III and IV.
E. LAPLACE TRANSFORM SOLUTION FOR THE SONIC WIND TUNNEL WALL INTERFERENCE PROBLEM

A parallel development to the collocation technique is found in the Laplace transform solution of the sonic wind tunnel wall interference problem [Ref. 4].

Figure 7. SONIC WIND TUNNEL

Figure 7 shows the arrangement of a blade facing a solid wall. Due to symmetry, the solution for the flow between the blade and the wall will be the same as that which would exist between the blade and the virtual blade beyond the wall. Hence it is the same solution expected of an unstaggered cascade with \( \pi \) phase shift between the blades.

The basic equation for oscillatory sonic flow from Section II is

\[
\psi_{yy} - Q\psi_x + K^2\psi = 0. \quad (II-49)
\]
The boundary-value problem can be formulated as follows:

1. There is no disturbance upstream of the blade leading edge.

2. Flow tangency conditions must be satisfied along the blade and along the solid wall.

This latter condition may be stated mathematically,

\[ \psi_y(X,0) = -v(X) \quad \text{at } Y = 0 \quad (III-48) \]

and

\[ \psi_y(X,d) = 0 \quad \text{at } Y = \pm d \quad (III-49) \]

Denoting the Laplace transformation of a function with respect to X by an overbar, i.e.,

\[ \mathcal{L}(f(X)) = \int_0^\infty e^{-pX} f(X) \, dX \]

one obtains for the Laplace transform of Eqn. (II-49)

\[ \overline{\psi}_{yy} - \lambda^2 \overline{\psi} = 0 \quad (III-50) \]

where

\[ \lambda^2 = Qp - k^2 \quad (III-51) \]

The boundary conditions transform into

\[ \overline{\psi}_y = -\overline{v}(p) \quad Y = 0 \quad (III-52) \]

and

\[ \overline{\psi}_y = 0 \quad Y = \pm d \quad (III-53) \]

A solution to (III-50) satisfying (III-52) and (III-53) has been given by Miles [Ref. 19],

\[ \overline{\psi}(p,Y) = \frac{1}{\lambda} \overline{v}(p)\text{C SCH}(\lambda d)\text{COSH}[\lambda(d-|Y|)]\text{sgn}(Y) \quad (III-54) \]
Restricting oneself to the flow between the upper tunnel wall $Y = d$ and the blade one has $Y \geq 0$. Hence the absolute value sign may be dropped.

To invert (III-54), a suitable expansion of the equation was given by Miles as

$$\Phi(p, Y) = \frac{1}{\lambda} \overline{v}(p) \sum_{N=0}^{\infty} \left( \text{EXP}[\lambda(2N+Y)] + \text{EXP}[\lambda((2N+2)d-Y)] \right)$$

(III-55)

Recalling the Laplace transform of

$$f(X) = x^{-1/2} e^{-\alpha/4X}$$

being

$$\overline{f}(p) = \int_{0}^{\infty} e^{-pX} f(X) \, dX = \sqrt{\pi} \frac{e^{-\sqrt{p\lambda}}}{\sqrt{p}} , \text{ Re } p > 0 ,$$

(III-56)

one has for

$$\overline{f}(p, Y) = \frac{1}{\lambda} \text{EXP}[\lambda(2Nd+Y)]$$

where $\lambda = \sqrt{Qs-K^2}$

the inversion

$$f(X, Y) = - \frac{\text{EXP}[\frac{K^2}{Q} X - \frac{Q}{4} (2Nd+Y)^2]}{\sqrt{\pi Q X}} .$$

(III-57)

Therefore, with the aid of the convolution theorem, one obtains the following solution:

$$\psi(X, Y) = - \frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \text{EXP}[\frac{K^2}{Q}(X-S)] \cdot$$

\[ \cdot \sum_{N=0}^{\infty} \left\{ \text{EXP}[\frac{Q}{4} \frac{(2Nd+Y)^2}{X-S}] + \text{EXP}[\frac{Q}{4} \frac{(2N+2)d+Y)^2}{X-S}] \right\} \, dS . \]

(III-58)
On the blade surface where \( Y = 0 \), this result simplifies to

\[
\psi(X,0) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \exp\left[\frac{K^2}{Q} (X-S)\right] \cdot \\
\cdot \sum_{N=0}^{\infty} \left\{ \exp\left[-\frac{Q N^2 d^2}{X-S} \right] + \exp\left[-\frac{Q}{4} \frac{(2N+2)^2 d^2}{X-S} \right] \right\} dS.
\]

(III-59)

For steady flow where \( K = 0 \), one obtains the further simplification,

\[
\psi(X,0) = -\frac{1}{\sqrt{\pi \lambda}} \left\{ \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} dS \\
+ 2 \sum_{N=1}^{\infty} \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \exp\left[-\lambda \frac{N^2 d^2}{X-S} \right] dS \right\}.
\]

(III-60)

This expression was previously given by Sandeman [Ref. 4] in his analysis of choked wind tunnel flow. Equation (III-58) represents the generalization to oscillatory flow.

Appendix B-II describes the translation of (III-60) into a computer program. The solution of this function provides the steady state check of the collocation technique as is shown in Tables III and IV.
IV. PRESENTATION OF RESULTS

Table II presents the interference coefficients for the steady state case obtained for the controlling parameters indicated in the tables with the computer routine developed in Appendix A.

Table III contains the value of the perturbation potential along the reference blade obtained from the routine developed in Appendix B-I and utilizing the interference coefficients of Table II. These results may be compared with those in Table IV obtained from the steady state Laplace transform solution for the sonic wind tunnel wall interference problem of Section III-E and programmed in Appendix B-II.

Table V demonstrates the erroneous correction integrals obtained from Subroutine COR of Appendix A as the result of computer numerical inaccuracies. A comparison of the interference coefficients for two cases before and after the anomaly was detected is presented in Table VI.

A sample of the unsteady interference coefficients for the case \( \lambda = 0 \) is presented in Table VII.

Table VIII compares a sample steady state output with the output of the general case \( \lambda \neq 0, K \neq 0 \), for very small \( K \).
<table>
<thead>
<tr>
<th>DL = 0.200</th>
<th>DK = 0.0</th>
<th>RHO = 1.000</th>
<th>AL = 0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEL(00) = (-8.0229E-01, 0.0)</td>
<td>DEL(10) = (2.0453E-00, 0.0)</td>
<td>DEL(01) = (-1.2504E-00, 0.0)</td>
<td>DEL(11) = (1.2504E-00, 0.0)</td>
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<tr>
<td>DEL(02) = (3.0619E-01, 0.0)</td>
<td>DEL(12) = (-3.0619E-01, 0.0)</td>
<td>DEL(03) = (-1.5100E-01, 0.0)</td>
<td>DEL(13) = (1.5100E-01, 0.0)</td>
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<tr>
<td>DEL(04) = (1.0639E-01, 0.0)</td>
<td>DEL(14) = (-1.0639E-01, 0.0)</td>
<td>DEL(05) = (-6.7890E-02, 0.0)</td>
<td>DEL(15) = (6.7890E-02, 0.0)</td>
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<tr>
<td>DEL(06) = (-2.0573E-02, 0.0)</td>
<td>DEL(16) = (2.0573E-02, 0.0)</td>
<td>DEL(07) = (3.3798E-03, 0.0)</td>
<td>DEL(17) = (-3.3798E-03, 0.0)</td>
</tr>
<tr>
<td>DEL(08) = (1.3068E-01, 0.0)</td>
<td>DEL(18) = (-1.3068E-01, 0.0)</td>
<td>DEL(09) = (-9.7604E-02, 0.0)</td>
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</tbody>
</table>

<table>
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| DEL(01) = (-8.5002E-03, 0.0) | DEL(11) = (8.5002E-03, 0.0) |
| DEL(02) = (-1.4889E-02, 0.0) | DEL(12) = (1.4889E-02, 0.0) |
| DEL(03) = (-5.3168E-03, 0.0) | DEL(13) = (5.3168E-03, 0.0) |
| DEL(04) = (7.0907E-03, 0.0) | DEL(14) = (-7.0907E-03, 0.0) |
| DEL(05) = (-6.0284E-04, 0.0) | DEL(15) = (6.0284E-04, 0.0) |
| DEL(06) = (-4.9886E-03, 0.0) | DEL(16) = (4.9886E-03, 0.0) |
| DEL(07) = (3.8379E-03, 0.0) | DEL(17) = (-3.8379E-03, 0.0) |
| DEL(08) = (1.6985E-03, 0.0) | DEL(18) = (-1.6985E-03, 0.0) |
| DEL(09) = (-2.1847E-03, 0.0) | DEL(19) = (2.1847E-03, 0.0) |

**TABLE II**  INTERFERENCE FUNCTION COEFFICIENTS

\[ DL = 0.200 \quad DK = 0.0 \quad \text{RHO} = 5.000 \quad \text{AL} = 0.0 \]

\[ DL = 0.200 \quad DK = 0.0 \quad \text{RHO} = 10.000 \quad \text{AL} = 0.0 \]
POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS
DL = 0.200  DK = 0.0  RHO = 1.000

THE INTERFERENCE COEFFICIENTS

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<tr>
<th>DEL(00)</th>
<th>DEL(01)</th>
<th>DEL(02)</th>
<th>DEL(03)</th>
<th>DEL(04)</th>
<th>DEL(05)</th>
<th>DEL(06)</th>
<th>DEL(07)</th>
<th>DEL(08)</th>
<th>DEL(09)</th>
<th>DEL(10)</th>
<th>DEL(11)</th>
<th>DEL(12)</th>
<th>DEL(13)</th>
<th>DEL(14)</th>
<th>DEL(15)</th>
<th>DEL(16)</th>
<th>DEL(17)</th>
<th>DEL(18)</th>
<th>DEL(19)</th>
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<td>-2.0453110E 00</td>
<td>-1.2504177E 00</td>
<td>3.0618846E-01</td>
<td>-1.5100145E-01</td>
<td>1.0638958E-01</td>
<td>-6.7890167E-02</td>
<td>2.0572830E-02</td>
<td>3.3797889E-03</td>
<td>1.3067639E-01</td>
<td>-9.7604275E-02</td>
<td>2.0453110E 00</td>
<td>1.2504177E 00</td>
<td>3.0618846E-01</td>
<td>1.5100145E-01</td>
<td>1.0638958E-01</td>
<td>6.7890167E-02</td>
<td>2.0572830E-02</td>
<td>3.3797889E-03</td>
<td>1.3067639E-01</td>
<td>9.7604275E-02</td>
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VALUE OF THE POTENTIAL ALONG THE BLADE

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<tr>
<th>STATION</th>
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</tr>
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<td>POT = 1.1780910E 00</td>
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<td>X = -0.80</td>
<td>POT = 2.1757832E 00</td>
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<td>X = -0.70</td>
<td>POT = 3.1754112E 00</td>
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<td>POT = 5.1756020E 00</td>
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<td>X = -0.40</td>
<td>POT = 6.1755772E 00</td>
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<td>X = -0.30</td>
<td>POT = 7.1755638E 00</td>
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<tr>
<td>X = -0.20</td>
<td>POT = 8.1755724E 00</td>
</tr>
<tr>
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<td>POT = 1.3175570E 01</td>
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<td>X = 0.60</td>
<td>POT = 1.6175568E 01</td>
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<td>X = 0.70</td>
<td>POT = 1.7175552E 01</td>
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<td>X = 0.90</td>
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<td>POT = 2.0176529E 01</td>
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TABLE III
POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS

\( DL = 0.200 \quad DK = 0.0 \quad RHO = 2.000 \)

THE INTERFERENCE COEFFICIENTS

\[
\begin{align*}
DE(00) &= -8.0229187E-01 \\
DE(01) &= -6.1007094E-01 \\
DE(02) &= 1.4253497E-01 \\
DE(03) &= -6.6098273E-02 \\
DE(04) &= 2.7520720E-02 \\
DE(05) &= -1.2785759E-02 \\
DE(06) &= 5.6872237E-02 \\
DE(07) &= -5.7169359E-02 \\
DE(08) &= -4.0132940E-02 \\
DE(09) &= 5.0579518E-02 \\
DE(10) &= 8.0229187E-01 \\
DE(11) &= 6.1007094E-01 \\
DE(12) &= -1.4253497E-01 \\
DE(13) &= 6.6098273E-02 \\
DE(14) &= -2.7520720E-02 \\
DE(15) &= 1.2785759E-02 \\
DE(16) &= -5.6872237E-02 \\
DE(17) &= 5.7169359E-02 \\
DE(18) &= 4.0132940E-02 \\
DE(19) &= -5.0579518E-02
\end{align*}
\]

VALUE OF THE POTENTIAL ALONG THE BLADE

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<thead>
<tr>
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<th>POTENTIAL</th>
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<tbody>
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<tr>
<td>X=-0.80</td>
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<tr>
<td>X=-0.70</td>
<td>POT= 1.8202248E 00</td>
</tr>
<tr>
<td>X=-0.60</td>
<td>POT= 2.3200750E 00</td>
</tr>
<tr>
<td>X=-0.50</td>
<td>POT= 2.8200855E 00</td>
</tr>
<tr>
<td>X=-0.40</td>
<td>POT= 3.3201075E 00</td>
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<tr>
<td>X=-0.30</td>
<td>POT= 3.8201075E 00</td>
</tr>
<tr>
<td>X=-0.20</td>
<td>POT= 4.3200874E 00</td>
</tr>
<tr>
<td>X=-0.10</td>
<td>POT= 4.8200750E 00</td>
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<td>X= 0.10</td>
<td>POT= 5.8200798E 00</td>
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<td>X= 0.20</td>
<td>POT= 6.3200665E 00</td>
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<tr>
<td>X= 0.30</td>
<td>POT= 6.8200712E 00</td>
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<td>X= 0.40</td>
<td>POT= 7.3200731E 00</td>
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<td>X= 0.50</td>
<td>POT= 7.8200769E 00</td>
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<td>POT= 8.3200636E 00</td>
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<td>POT= 9.3201199E 00</td>
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<td>X= 0.90</td>
<td>POT= 9.8202019E 00</td>
</tr>
<tr>
<td>X= 1.00</td>
<td>POT= 1.0319364E 01</td>
</tr>
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</table>

| TABLE III |

58
POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS
DL= 0.200      DK= 0.0      RHO= 5.000

THE INTERFERENCE COEFFICIENTS

\[
\begin{align*}
\text{DEL}(00) &= -1.1538456 \times 10^{-01} \\
\text{DEL}(01) &= -1.8925136 \times 10^{-01} \\
\text{DEL}(02) &= 7.1354136 \times 10^{-03} \\
\text{DEL}(03) &= 2.2944096 \times 10^{-02} \\
\text{DEL}(04) &= -2.5751479 \times 10^{-02} \\
\text{DEL}(05) &= 2.8709877 \times 10^{-02} \\
\text{DEL}(06) &= -6.2098347 \times 10^{-02} \\
\text{DEL}(07) &= 4.1337617 \times 10^{-02} \\
\text{DEL}(08) &= 5.2630220 \times 10^{-02} \\
\text{DEL}(09) &= -5.1547259 \times 10^{-02} \\
\text{DEL}(10) &= 1.1538456 \times 10^{-01} \\
\text{DEL}(11) &= 1.8925136 \times 10^{-01} \\
\text{DEL}(12) &= -7.1354136 \times 10^{-03} \\
\text{DEL}(13) &= -2.2944096 \times 10^{-02} \\
\text{DEL}(14) &= 2.5751479 \times 10^{-02} \\
\text{DEL}(15) &= 2.8709877 \times 10^{-02} \\
\text{DEL}(16) &= 6.2098347 \times 10^{-02} \\
\text{DEL}(17) &= 4.1337617 \times 10^{-02} \\
\text{DEL}(18) &= 5.2630220 \times 10^{-02} \\
\text{DEL}(19) &= 5.1547259 \times 10^{-02}
\end{align*}
\]

VALUE OF THE POTENTIAL ALONG THE BLADE

<table>
<thead>
<tr>
<th>STATION</th>
<th>POTENTIAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>X=-0.90</td>
<td>POT= 7.9726553E-01</td>
</tr>
<tr>
<td>X=-0.80</td>
<td>POT= 1.1282368E 00</td>
</tr>
<tr>
<td>X=-0.70</td>
<td>POT= 1.3855448E 00</td>
</tr>
<tr>
<td>X=-0.60</td>
<td>POT= 1.6115170E 00</td>
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<tr>
<td>X=-0.50</td>
<td>POT= 1.8232851E 00</td>
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<tr>
<td>X=-0.40</td>
<td>POT= 2.0286140E 00</td>
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<td>POT= 2.2310429E 00</td>
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<td>POT= 4.4330502E 00</td>
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TABLE III
POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS

DL = 0.200  \quad DK = 0.0  \quad RHO = 10.000

THE INTERFERENCE COEFFICIENTS

\begin{align*}
 DL(00) &= -1.5652659E-03 \\
 DL(01) &= -8.5002407E-03 \\
 DL(02) &= -1.4888730E-02 \\
 DL(03) &= -5.3168200E-03 \\
 DL(04) &= 7.0907287E-03 \\
 DL(05) &= 6.0283695E-04 \\
 DL(06) &= -4.9886145E-03 \\
 DL(07) &= 3.8378909E-03 \\
 DL(08) &= 1.6985249E-03 \\
 DL(09) &= -2.1847319E-03
\end{align*}

\begin{align*}
 DL(10) &= 1.5652659E-03 \\
 DL(11) &= 8.5002407E-03 \\
 DL(12) &= 1.4888730E-02 \\
 DL(13) &= 5.3168200E-03 \\
 DL(14) &= -7.0907287E-03 \\
 DL(15) &= 6.0283695E-04 \\
 DL(16) &= 4.9886145E-03 \\
 DL(17) &= -3.8378909E-03 \\
 DL(18) &= 1.6985249E-03 \\
 DL(19) &= 2.1847319E-03
\end{align*}

VALUE OF THE POTENTIAL ALONG THE BLADE

<table>
<thead>
<tr>
<th>STATION</th>
<th>POTENTIAL</th>
</tr>
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<tbody>
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<td>X=-0.90</td>
<td>POT= 7.9786545E-01</td>
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<tr>
<td>X=-0.80</td>
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TABLE III
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<td>5.16666663E 00</td>
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<tr>
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<td>x = 1.00</td>
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</table>

**TABLE IV**

VALUE OF THE POTENTIAL ALONG THE BLADE--WALL INTERFERENCE TECHNIQUE

DL = 0.200
DK = 0.0

\[ \rho = 1.00 \]
### Table IV  Value of the Potential Along the Blade-Wall Interference Technique

**Solution for the Following Parameters**

- **DL** = 0.200
- **DK** = 0.0
- **RHO** = 2.000

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<th>Station</th>
<th>Potential</th>
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<tr>
<td>X = -0.80</td>
<td>POT = 1.33332E 00</td>
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<tr>
<td>X = -0.70</td>
<td>POT = 1.83333E 00</td>
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<tr>
<td>X = -0.60</td>
<td>POT = 2.33333E 00</td>
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<tr>
<td>X = -0.50</td>
<td>POT = 2.83333E 00</td>
</tr>
<tr>
<td>X = -0.40</td>
<td>POT = 3.33333E 00</td>
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<tr>
<td>X = -0.30</td>
<td>POT = 3.83333E 00</td>
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<tr>
<td>X = -0.20</td>
<td>POT = 4.33333E 00</td>
</tr>
<tr>
<td>X = -0.10</td>
<td>POT = 4.83333E 00</td>
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<td>X = -0.00</td>
<td>POT = 5.33332E 00</td>
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<td>X =  0.10</td>
<td>POT = 5.83333E 00</td>
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<tr>
<td>X =  0.20</td>
<td>POT = 6.33332E 00</td>
</tr>
<tr>
<td>X =  0.30</td>
<td>POT = 6.83331E 00</td>
</tr>
<tr>
<td>X =  0.40</td>
<td>POT = 7.33331E 00</td>
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<tr>
<td>X =  0.50</td>
<td>POT = 7.83332E 00</td>
</tr>
<tr>
<td>X =  0.60</td>
<td>POT = 8.33331E 00</td>
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<tr>
<td>X =  0.70</td>
<td>POT = 8.83331E 00</td>
</tr>
<tr>
<td>X =  0.80</td>
<td>POT = 9.33331E 00</td>
</tr>
<tr>
<td>X =  0.90</td>
<td>POT = 9.83332E 00</td>
</tr>
<tr>
<td>X =  1.00</td>
<td>POT = 1.03333E 01</td>
</tr>
</tbody>
</table>
TABLE IV  VALUE OF THE POTENTIAL ALONG THE BLADE--WALL INTERFERENCE TECHNIQUE

SOLUTION FOR THE FOLLOWING PARAMETERS

\[ DL = 0.200 \quad DK = 0.0 \quad RHO = 5.000 \]

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<thead>
<tr>
<th>STATION</th>
<th>POTENTIAL</th>
</tr>
</thead>
<tbody>
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<td>POT= 1.12867E 00</td>
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<td>X=-0.70</td>
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<td>POT= 1.61180E 00</td>
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<tr>
<td>X=-0.50</td>
<td>POT= 1.82356E 00</td>
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<tr>
<td>X=-0.40</td>
<td>POT= 2.02889E 00</td>
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<tr>
<td>X=-0.30</td>
<td>POT= 2.23132E 00</td>
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<tr>
<td>X=-0.20</td>
<td>POT= 2.43242E 00</td>
</tr>
<tr>
<td>X=-0.10</td>
<td>POT= 2.63291E 00</td>
</tr>
<tr>
<td>X=-0.00</td>
<td>POT= 2.83314E 00</td>
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<tr>
<td>X= 0.10</td>
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<td>X= 1.00</td>
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TABLE IV: VALUE OF THE POTENTIAL ALONG THE BLADE-WALL INTERFERENCE TECHNIQUE
SOLUTIONS FOR THE FOLLOWING PARAMETERS

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<tr>
<th>DL</th>
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DL=0. CORRECTION USED

SIN AND COS FRESNEL INTEGRAL VALUE FOR UPPER LIMIT AR.

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<th>AR</th>
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</thead>
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<tr>
<td>0.38822D 00</td>
<td>0.48815D 00</td>
<td>0.12500D 02</td>
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Solution to integral \((x^{*(j-2.5)}\sin\text{ or }\cos(\text{E}/\text{X})\text{ on limits zero to }\text{AL}=5.000D-03\text{ and for E=6.250D-02})

<table>
<thead>
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<th>J</th>
<th>SIN INTEGRAL</th>
<th>COS INTEGRAL</th>
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</thead>
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<td>2</td>
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<td>1.0112D-03</td>
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<tr>
<td>3</td>
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<td>6.9534D-06</td>
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<tr>
<td>4</td>
<td>1.2694D-07</td>
<td>4.3037D-08</td>
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<tr>
<td>5</td>
<td>6.0104D-10</td>
<td>2.5309D-10</td>
</tr>
<tr>
<td>6</td>
<td>2.8637D-12</td>
<td>1.4515D-12</td>
</tr>
<tr>
<td>7</td>
<td>1.3830D-14</td>
<td>7.5455D-15</td>
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<tr>
<td>8</td>
<td>1.9751D-16</td>
<td>1.1803D-16</td>
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<td>9</td>
<td>1.9863D-16</td>
<td>-3.2614D-16</td>
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<tr>
<td>10</td>
<td>4.5313D-15</td>
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<td>11</td>
<td>-1.6338D-14</td>
<td>2.6829D-14</td>
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<td>12</td>
<td>-6.1534D-15</td>
<td>-3.7471D-15</td>
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<tr>
<td>13</td>
<td>1.4279D-14</td>
<td>-2.3449D-14</td>
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SOLUTIONS FOR THE FOLLOWING PARAMETERS

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<td>5.000</td>
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DL=0, CORRECTION USED
SIN AND COS FRENSNEL INTEGRAL VALUE FOR UPPER LIMIT AR.

<table>
<thead>
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<th>COS INTEGRAL</th>
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</thead>
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<td>-2.7898D 05</td>
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<td>13</td>
<td>-7.4507D 07</td>
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SOLUTION TO INTEGRAL \( J - 2.5 \) SIN OR COS \( \frac{E}{X} \) ON LIMITS ZERO TO \( AL = 5.000D-02 \) AND FOR \( E = 6.250D 03 \)
### Interference Function Coefficients

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<th>RHO =  0.500</th>
<th>AL  =  0.0</th>
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<tr>
<td>DEL(01) = (-9.3527E-01, 7.7419E-01)</td>
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<tr>
<td>DEL(02) = (-6.8607E-02, -4.4571E-01)</td>
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<tr>
<td>DEL(03) = (-2.9892E-02, 1.8759E-01)</td>
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<tr>
<td>DEL(04) = (3.8507E-02, -1.2267E-01)</td>
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<tr>
<td>DEL(05) = (-3.3817E-02, 6.7233E-02)</td>
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<tr>
<td>DEL(06) = (-5.7924E-02, 3.1015E-02)</td>
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<tr>
<td>DEL(07) = (6.4039E-02, 1.6917E-02)</td>
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<tr>
<td>DEL(08) = (-9.6658E-02, -1.7150E-01)</td>
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<tr>
<td>DEL(09) = (-9.7952E-02, 1.1401E-01)</td>
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<table>
<thead>
<tr>
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<th>RHO =  0.500</th>
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<tr>
<td>DEL(12) = (6.8607E-02, 4.4571E-01)</td>
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<tr>
<td>DEL(13) = (2.9892E-02, 1.8759E-01)</td>
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#### Table VIA Interference Coefficients Before and After Error Detection
### INTERFERENCE FUNCTION COEFFICIENTS

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<th>AL</th>
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<td>5.000</td>
<td>0.0</td>
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<td>( \text{DEL}(0) = (9.8874 \times 10^{-1}, 2.3480 \times 10^{-3}) )</td>
<td>( \text{DEL}(10) = (-9.8874 \times 10^{-1}, -2.3480 \times 10^{-3}) )</td>
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<td></td>
</tr>
<tr>
<td>( \text{DEL}(01) = (-9.3834 \times 10^{-3}, 5.0453 \times 10^{-3}) )</td>
<td>( \text{DEL}(11) = (9.3834 \times 10^{-3}, -5.0453 \times 10^{-3}) )</td>
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</tr>
<tr>
<td>( \text{DEL}(02) = (-2.5635 \times 10^{-3}, -4.6483 \times 10^{-4}) )</td>
<td>( \text{DEL}(12) = (2.5635 \times 10^{-3}, 4.6483 \times 10^{-4}) )</td>
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<tr>
<td>( \text{DEL}(03) = (-1.3289 \times 10^{-3}, -9.1403 \times 10^{-3}) )</td>
<td>( \text{DEL}(13) = (1.3289 \times 10^{-3}, 9.1403 \times 10^{-3}) )</td>
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<tr>
<td>( \text{DEL}(04) = (1.3552 \times 10^{-2}, -2.9720 \times 10^{-2}) )</td>
<td>( \text{DEL}(14) = (-1.3552 \times 10^{-2}, 2.9720 \times 10^{-2}) )</td>
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<tr>
<td>( \text{DEL}(05) = (4.2880 \times 10^{-1}, 1.2744 \times 10^{-1}) )</td>
<td>( \text{DEL}(15) = (-4.2880 \times 10^{-1}, -1.2744 \times 10^{-1}) )</td>
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<tr>
<td>( \text{DEL}(06) = (-1.4891 \times 10^{-2}, 3.9460 \times 10^{-3}) )</td>
<td>( \text{DEL}(16) = (1.4891 \times 10^{-2}, -3.9460 \times 10^{-3}) )</td>
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<tr>
<td>( \text{DEL}(07) = (-2.4377 \times 10^{-3}, -6.0711 \times 10^{-3}) )</td>
<td>( \text{DEL}(17) = (2.4377 \times 10^{-3}, 6.0711 \times 10^{-3}) )</td>
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<tr>
<td>( \text{DEL}(08) = (6.2252 \times 10^{-3}, -1.5752 \times 10^{-3}) )</td>
<td>( \text{DEL}(18) = (-6.2252 \times 10^{-3}, 1.5752 \times 10^{-3}) )</td>
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<tr>
<td>( \text{DEL}(09) = (-3.6667 \times 10^{-6}, -1.4728 \times 10^{-5}) )</td>
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### TABLE VIB  INTERFERENCE COEFFICIENTS BEFORE AND AFTER ERROR DETECTION
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<tr>
<td>DEL(02) = (-6.8607E-02, -4.4571E-01)</td>
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<td>DEL(03) = (-2.9892E-02, 1.8759E-01)</td>
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<td>DEL(04) = ( 3.8507E-02, -1.2267E-01)</td>
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<td>DEL(07) = ( 6.4039E-02, 1.6914E-02)</td>
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<tr>
<td>DEL(02) = (-5.8491E-02, -2.2545E-01)</td>
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<tr>
<td>DEL(03) = ( 1.7249E-02, 9.6792E-02)</td>
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<tr>
<td>DEL(06) = (-1.4476E-01, 3.0627E-01)</td>
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<td>DEL(12) = ( 5.8491E-02, 2.2545E-01)</td>
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<td>DEL(13) = ( -1.7249E-02, -9.6792E-02)</td>
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<td>DEL(15) = ( 1.2356E-01, -1.2207E-01)</td>
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<td>DEL(16) = ( 1.4476E-01, -3.0627E-01)</td>
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<tr>
<td>DEL(17) = ( -3.6416E-01, 3.2108E-01)</td>
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<tr>
<td>DEL(18) = ( -9.2039E-02, 4.4756E-01)</td>
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<td>DL</td>
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**TABLE VII**

INTERFERENCE FUNCTION COEFFICIENTS
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DEL(05)=(-5.0269E 01, -1.0972E 02)  DEL(15)=( 5.0269E 01,  1.0972E 02)  
DEL(06)=( 8.6721E 01, -2.7566E 00)  DEL(16)=(-8.6721E 01,  2.7566E 00)  
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DEL(08)=(-4.9922E 01, -2.5936E 00)  DEL(18)=( 4.9922E 01,  2.5936E 00)  
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**TABLE VIII**

**INTERFERENCE FUNCTION COEFFICIENTS**

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A computer program was designed to compute interference coefficients for the collocation technique developed in Section III-A. For the steady case, the resulting coefficients in Table II were applied to the technique developed in Section III-C and III-D to provide the values of the blade surface potentials in Table III. For this same case, the sonic wind tunnel wall interference solution of Section III-E provided the data presented in Table IV which verified the collocation technique for that case.

Although no "separate source" checks were made for the coefficients presented in Table VII (The case of simple oscillation about the mid-chord with no translation.), every effort was made to verify the procedure. Table VIII compares the proven steady state solution with a general small K solution. Unsteady integral approximation techniques were justified in Sections III-B-1, -2, and -3 as summarized below.

Difficulties arising from the solution of the occurring integrals in the unsteady solution were not insurmountable but required careful handling. Whenever numerical integration of functions involving $\frac{1}{X}$ or $\sin\left(\frac{1}{X}\right)$ near $X = 0$ is attempted, accuracy is suspect. One must closely inspect arguments, limits, and results to ensure validity. For example, for the unsteady case and $\lambda \neq 0$, with no apparent
means of computing the integral in the immediate vicinity of the origin, one must decide, against the shortcomings in computer capacity and programming methods, just what approximation must be applied and, once applied, how accurate it is. In this case, the question arose as to how closely one might approach the origin with numerical integration techniques before accuracy was degraded. At the same time the required computer time increased dramatically as the integrand became more ill-behaved.

The most difficult question that arose in this investigation was the accuracy of the correction integral in the case \( \lambda = 0 \). Once this approach was found valid for \( J = 1 \) through the calculations shown in Section III-B-2, there remained the need to improve the accuracy of the technique and ensure its validity for \( J > 1 \). In question were the number of terms needed in the truncated series shown in Equations (III-40) and (III-41), the validity of the integral reduction formula used in subroutine RED, and the selection of the upper limit of the correction integral.

Although four terms were used in the truncated series for the case \( J = 1 \) (Refer to Appendix A-IV), only three terms affected the significant figures of the results for the parameters used in this investigation. For \( J = 2, 3, \) and \( 4 \), the number of terms used was decreased by one for each computation. The primary motivation for this step was the inaccuracy of the integral reduction formula for large \( J \) and large \( R \).
Referring to Tables VA and VB, it was determined that numerical inaccuracies began to occur for large \( J \) in the integral reduction routine. The effect of this inaccuracy was not important for small \( R \) but for the case of \( R = 5 \), for example, it overwhelmed the results. For integrals of the form

\[
\int_{0}^{AL} U^{J-\frac{1}{2}} \sin(E/U) \cos(U) dU,
\]

as \( J \) becomes large and for \( AL << 1 \), the value of the integral must become small. However, in the case of \( R = 0.5 \), the magnitude of the integral values diminished with increasing \( J \) up to \( J = 9 \) but increased for \( J > 9 \). In the case of \( R = 5 \), the anomaly in the computation affected integrals for \( J \)'s as low as five. (In this latter case, it is obvious that the erroneous answer for \( J = 13 \) must affect the results.)

The cause of this error was determined to be differences taken between large numbers in the reduction formula. The value obtained for the Fresnel integrals was only as accurate as the curve fit used in Subroutine CS. This error and the error generated in computing the SIN and COS functions occurring in the formula resulted in erroneous answers of a magnitude that was directly related to \( J \) and \( R \). To correct the problem, the use of the integral reduction routine for \( J \geq 5 \) was eliminated.

For \( J \geq 5 \), numerical techniques were resorted to. Here, for \( AL << 1 \), the amplitude of the integrand was small.
permitting one to neglect that portion of the integral near zero. This technique seemed to produce reasonable results, at least the results did not have large magnitudes that would have affected the final answers. The resulting integral values were in some cases less than the convergence limits placed on the numerical integration subroutine, ZSUM, and were, therefore, assumed to contain error not greater than the said limits. Further, as these integral values became small, it should have been possible to neglect them altogether since the values of those integrals with small \((J)\) predominated in the series.

Comparative results for the case \(K = 0.5\) and \(R = 0.5\) and 5.0 are presented before and after the error was detected in Tables VIA and VIB.

The upper limit of the correction integral was selected such that the approximation
\[
\frac{\sin(E/U)}{\cos(DU + E/U)} = \frac{\sin(DU)}{\cos(U)}
\]
valid in the range \(0 \leq U \leq AL\). The point \(AL\) also needed to be within the capabilities of the numerical integration scheme on the limits \(AL \leq U \leq X + 1\). Such a point was picked at \(AL = R/100\) which ensures that (For \(U \leq AL\)) \(E/U\) was at least \(1.0E4\) larger than \(DU\). Such an \(AL\) was also within the capabilities of the numerical technique over the range of parameters used in this investigation. Table IB demonstrates that the use of the truncated series in this approximation decreases its sensitivity to \(AL\).

Computer time could have been reduced with a larger \(AL\) if one were willing to accept reasonable inaccuracies as
the term DU became more significant. A smaller AL increased computer time and the accuracy of the numerical integration technique became more and more questionable as the integral became more ill-behaved. Computer time for a single run of the program (for $\lambda = 0$) on the IBM 360/67 computer system was on the order of 15 minutes.

It should be noted that in the absence of a good error measurement yardstick, statements concerning more or less error are indeed relative. It is difficult to assess where computer numerical error or approximation error becomes more important. It is for this reason that an independent check of these results against another source is necessary. The Laplace transform solution for the sonic unsteady wind tunnel wall interference problem would, in theory, be such a check; however, this technique will result in integrals of the same form as those used in the collocation solution and will require similar (or the same) solution techniques. It will be difficult, then, to obtain a completely independent check of these results.

Once verification or modification of results presented in this investigation is in hand, application of Spreiter's local linearization technique alluded to in Section II should be applicable and lead to reasonable analytic results for transonic pressure and moment distribution prediction.
VI. SUMMARY

Following the basic approach offered by Gorelov for subsonic and low supersonic flows, a collocation technique was applied to the purely transonic case superimposing the isolated foil solution provided by Teipel with unknown interference potentials. A computer routine was developed which would generate the coefficients for the unknown interference source distributions in the steady case. The interference coefficients in hand, the potential along the blade was determined through a second routine. These results were favorably compared with the potential solution derived from a routine based on the Laplace transform treatment of the sonic wind tunnel wall interference problem for the steady case. All three of these computer routines initially utilized series expansions which reduced the occurring integrals to functions of the well known error function. Confirmation of these results led to an extension of the collocation approach into the unsteady regime.

First attempts to utilize series approximations for the unsteady case resulted in failure due to nonconvergence of the series. Faced with a lack of an apparently simple solution to the resulting integrals, numerical integration was resorted to. Here difficulties arose centering around the pathological behavior of the integrands near the origin as pointed out in the Discussion of Results.
As a basic numerical integration technique, the "pyramided" Simpson's rule method was incorporated. For the steady case where the integrals were easily computed, the results of this method were checked against the steady state case already in hand and found to agree perfectly.

The greatest difficulty occurred in the unsteady case for $\lambda = 0$ where one could not reasonably neglect the integrals in the interval near zero. To overcome this problem an analytical approximation (for the integral evaluated near zero) was introduced and checked against a known analytical solution shown in Section III. The technique involved the reduction of the integrals to functions of the Fresnel integrals. However, this procedure resulted in additional problems which were not immediately detected. Although the technique was theoretically sound, the computer numerical error pointed out in the Discussion of Results became apparent after study of the output. Once the cause was detected, the anomaly was remedied in a simple manner.

The coefficients for the unsteady interference functions as presented here shall require a "separate source" check. The Laplace transform solution for the oscillatory transonic wind tunnel interference problem should provide such a check. If these techniques verify one another, a third independent check could be the Hamamoto linearized high frequency solution.

Development of required computer routines for the potential and pressure coefficients from the collocation
and Laplace techniques should be facilitated by the use of subroutines derived for the unsteady coefficient routine presented here.

It is felt that the collocation technique will prove a versatile tool in transonic flow studies in that the freedom which one has in specifying boundary conditions will permit treatment of problems which defy solution by other techniques. For example, the sonic wind tunnel wall interference technique is only good in the $\pi$ phase shift case while the collocation technique will provide results for any phase relationship. Further, it should be valid for more complex boundary conditions such as those posed by wind tunnels with porous walls.
APPENDIX A
FORMULATION OF THE COLLOCATION COEFFICIENT ROUTINE

A-I. MAIN PROGRAM

The purpose of the main program is the solution of Equations (III-16) and (III-17) for the unknown interference function coefficients.

\[
\begin{align*}
L-1 \sum_{N=0}^{L-1} \left\{ P_{ON} X^N + \frac{R}{2\pi} \left( \theta_{IN} + \theta_{ON} \right) \right\} \frac{X}{(X-S)^{3/2}} \int_{-1}^{1} \exp\left[ \frac{K^2}{Q} (X-S) - \frac{Q}{4} \frac{R^2}{X-S} \right] dS = 0 \\
(III-16)
\end{align*}
\]

\[
\begin{align*}
L-1 \sum_{N=0}^{L-1} \left\{ P_{ON} X^N + \frac{R}{2\pi} \left( \theta_{ON} + \theta_{IN} \right) \right\} \frac{X}{(X-S)^{3/2}} \int_{-1}^{1} \exp\left[ \frac{K^2}{Q} (X-S) - \frac{Q}{4} \frac{R^2}{X-S} \right] dS = 0 \\
(III-17)
\end{align*}
\]

Equation (III-16) must be satisfied for each station along the lower blade while simultaneously equation (III-17) must be satisfied at each station along the upper blade. For the purposes of this routine, ten stations have been selected \((X_r = -0.9 + 0.2r, r = 0, 1, 2, \ldots, 9)\). Computations near the leading edge are avoided due to the singularity there.

For a given \(K, \lambda, \text{and} R\), the term

\[
\frac{R}{2\pi} \int_{-1}^{1} \frac{X}{(X-S)^{3/2}} \exp\left[ \frac{K^2}{Q} (X-S) - \frac{Q}{4} \frac{R^2}{X-S} \right] dS = SR(X,N) + iSM(X,N)
\]

(A-I-1)
is a complex function of the parameters $X$ and $N$. The complex value of this function at each $(X,N)$ is provided by Subroutine WSUM and will be dealt with under Appendix A-II. WSUM returns the value of $K$ as real and imaginary parts in arrays $SR$ and $SM$ respectively.

The problem then is to evaluate the various known or computed values and insert them in proper form in a suitable matrix for simultaneous solution.

Observe (III-16) and (III-17) become

$$
\sum_{N=0}^{9} \left\{ \mathcal{D}_{ON} X^N + (\theta_{IN} + \mathcal{D}_{IN})(SR(X,N) + iSM(X,N)) \right\} = 0
$$
(A-I-2)

and

$$
\sum_{N=0}^{9} \left\{ \mathcal{D}_{IN} X^N + (\theta_{ON} + \mathcal{D}_{ON})(SR(X,N) + iSM(X,N)) \right\} = 0
$$
(A-I-3)

With $\theta_{IN}$ and $\theta_{ON}$ determined by boundary conditions at each of the ten stations, (A-I-2) and (A-I-3) form a series of twenty equations in unknowns $\mathcal{D}_{ON}$ and $\mathcal{D}_{IN}$, $N = 0, 1, 2, \ldots, 9$. Note, for the unsteady case, $\theta_{MN}$ and $\mathcal{D}_{MN}$ are complex.

Taking the set of coefficients, $\theta_{MN}$, from (III-21), (A-I-2) and (A-I-3) are rewritten for the in phase case as

$$
\sum_{N=0}^{9} \left\{ \mathcal{D}_{ON} X^N + \mathcal{D}_{IN}(SR(X,N)+iSM(X,N)) \right\} = (SR(X,0) - DK SM(X,1))
$$

$$
+ i(SM(X,0) + DK SR(X,1))
$$
(A-I-4)
and

\[
\sum_{N=0}^{9} \left\{ D_{IN}^X N + D_{ON}^X (SR(X,N) + iSM(X,N)) \right\} = (SR(X,0) - DK SM(X,1))
\]

\[+ i(SM(X,0) + DK SR(X,1)) \] (A-I-5)

where DK replaces K in the computer routine.

For the \( \pi \) phase shift case, the coefficients (III-20) apply and the lower blade boundary equation (A-I-2) becomes

\[
\sum_{N=0}^{9} \left\{ D_{ON}^X N + D_{IN}^X (SR(X,N) + iSM(X,N)) \right\} = -(SR(X,0) - DK SM(X,1))
\]

\[ - i(SM(X,0) + DK SR(X,1)) \] (A-I-6)

The system of twenty equations may be written in matrix form

\[
\bar{X} \bar{D} = \bar{B} \] (A-I-7)

where \( \bar{X} \), \( \bar{D} \), and \( \bar{B} \) are the matrices of coefficients, unknown \( D \)'s, and right hand sides. \( \bar{X} \), \( \bar{D} \), and \( \bar{B} \) are complex and may be written

\[
\bar{X} = \bar{X}_1 + \bar{X}_2 i
\]

\[
\bar{D} = \bar{D}_1 + \bar{D}_2 i \] (A-I-8)

and

\[
\bar{B} = \bar{B}_1 + \bar{B}_2 i.
\]
Then in (A-I-7)

\[(X_1 + X_2i)(\overline{D}_1 + \overline{D}_2i) = \overline{B}_1 + \overline{B}_2i\]

or

\[(\overline{D}_1X_1 - \overline{D}_2X_2) + i(\overline{D}_2X_1 + \overline{D}_1X_2) = \overline{B}_1 + \overline{B}_2i. \quad (A-I-9)\]

It may be concluded that

\[\overline{D}_1X_1 - \overline{D}_2X_2 = \overline{B}_1\]

\[\overline{D}_1X_2 + \overline{D}_2X_1 = \overline{B}_2\]

or

\[
\begin{pmatrix}
X_1 & -X_2 \\
X_2 & X_1
\end{pmatrix}
\begin{pmatrix}
\overline{D}_1 \\
\overline{D}_2
\end{pmatrix}
= 
\begin{pmatrix}
\overline{B}_1 \\
\overline{B}_2
\end{pmatrix}
\quad (A-I-10)
\]

The twenty complex equations in twenty unknowns represented by (A-I-7) have been transformed into 40 real equations in 40 real unknowns. One has only to solve for \(\overline{D}_1\) and \(\overline{D}_2\) and set

\[\overline{D} = \overline{D}_1 + i\overline{D}_2.\]

PROGRAM STEP ANALYSIS

(Refer to Routine MAIN at the end of this Appendix.)

Double precision is used throughout the routine for
numerical accuracy. The use of the various arrays and common storage areas that are declared will be detailed below.

Various arrays and fixed parameters are initialized in MAIN to minimize computational redundancy in later programming. The arrays COC(10), COS(10), AS(10), and AC(10), zeroized in DO LOOP 100, are utilized in Subroutine WSUM. DL = λ, DK = K, and RHO = R, the controlling parameters, are initialized and PI = π is set. The value for

\[ A = \frac{(DK)^2}{(DL)^2 + 4(DK)^2} \tag{A-I-11} \]

is computed. For the case DK = 0, A is set to zero to avoid possible computer problems involving division by zero. Additional parameters are computed as follows:

\[ B = \frac{DL(RHO)^2}{4} \tag{A-I-12} \]

\[ C = \frac{DL(DK)^2}{(DL)^2 + 4(DK)^2} \tag{A-I-13} \]

\[ D = \frac{2(DK)^3}{(DL)^2 + 4(DK)^2} \tag{A-I-14} \]

\[ E = \frac{DK(RHO)^2}{2} \tag{A-I-15} \]

\[ G = \frac{1}{2} \tan(\frac{2DK}{DL}) \tag{A-I-16} \]
Here $G$ is set to

$$G = \frac{\pi}{4}$$

(A-I-16a)

for the case $DL = 0$ to avoid division by zero.

$$AK = \sin(G)$$

(A-I-17)

$$BK = \cos(G)$$

(A-I-18)

$$Z = \frac{\text{RHO} \left[ (DL)^2 + 4(DK)^2 \right]}{2 \sqrt{PT}}$$

(A-I-19)

For future ease in changing the number of stations on the blades, the parameter $L$ is used. Here $L = 10$ is set. The initial station, $X_1 = -0.9$, and the interval between stations, $X_{\text{INT}} = 0.2$, are set.

The first write statement is included to indicate the controlling parameters $DL$, $DK$, and $\text{RHO}$.

Finally, the matrix manipulation section begins. Referencing (A-I-10), array $\text{AA}(40,40)$ is dimensioned to hold the coefficient matrix. $\text{BB}(40)$ holds the matrix of right hand sides before solution and the matrix of unknowns after solution. (A-I-10) reflects the diagonal duplication of the real and imaginary submatrices, $\bar{X}_1$ and $\bar{X}_2$. Note that equations (A-I-4) and (A-I-5) form adjacent rows in the array and that there is duplication between them for a given $X$. The first two rows of $\bar{X}_1$ are as follows:
\[
\begin{array}{cccccccccc}
1 & x_1 & \cdots & x_1^9 & SR(x_1,0) & SR(x_1,1) & \cdots & SR(x_1,9) \\
SR(x_1,0) & SR(x_1,1) & \cdots & SR(x_1,9) & 1 & x_1 & \cdots & x_1^9 \\
\end{array}
\]

(A-I-20)

corresponding to the unknowns

\[
\begin{array}{cccccccc}
\rho_{R00} & \rho_{R01} & \cdots & \rho_{R09} & \rho_{R10} & \rho_{R11} & \cdots & \rho_{R19}
\end{array}
\]

where \( \rho_{MN} = \rho_{Rmn} + i\rho_{Imn} \). The first two rows of \( \bar{X}_2 \) are:

\[
\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & SI(x_1,0) & SI(x_1,1) & \cdots & SI(x_1,9) \\
SI(x_1,0) & SI(x_1,1) & \cdots & SI(x_1,9) & 0 & 0 & \cdots & 0 \\
\end{array}
\]

(A-I-21)

corresponding to

\[
\begin{array}{cccccccc}
\rho_{I00} & \rho_{I01} & \cdots & \rho_{I09} & \rho_{I11} & \rho_{I12} & \cdots & \rho_{I19}
\end{array}
\]

Reference to (A-I-10) shows the placement of these rows.
The first row of \( \bar{X}_1 \) fills AA(1,1) to AA(1,20) as well as AA(21,21) to AA(21,40). The first row of \( \bar{X}_2 \) fills AA(1,21) to AA(1,40), and, with signs reversed, AA(21,1) to AA(21,20).
For the \( \pi \) phase shift case, the right hand side matrix is filled four rows at a time as follows:
BB(1) = -(SR(X_1,0) - DK SI(X_1,1))
BB(2) = (SR(X_1,0) - DK SI(X_1,1))
BB(21) = -(SI(X_1,0) + DK SR(X_1,1))
BB(22) = (SI(X_1,0) + DK SR(X_1,1))

and

BB(3) = -(SR(X_2,0) - DK SI(X_2,1))
BB(4) = (SR(X_2,0) - DK SI(X_2,1))
BB(23) = -(SI(X_2,0) + DK SR(X_2,1))
BB(24) = (SI(X_2,0) + DK SR(X_2,1))

and so forth.

DO LOOP 1000 on J = 1,L' varies the station from X_1 to X_{10}. It sets two rows in submatrices X_1 and X_2, or, equivalently, four rows in AA and four in BB. A is used to set the power of X in submatrix X_1 and is initially set to A = X^0 = 1. JJ is a counter used to select every other row, i.e., JJ = 1,3,...,19 as J = 1,2,...,10.

With X_1 set at X = -0.9, WSUM is called to provide the integral term in the boundary equations. WSUM computes the integral for X_1 and for N = 0,1,2,...,9. These ten values are returned in common arrays SR and SM. SR holds the real parts and SM the imaginary as shown in functional notation in (A-I-1). The arrays are filled as follows:
SR(1) = SR(X,0)    SM(1) = SM(X,0)  
SR(2) = SR(X,1)    SM(2) = SM(X,1)  
    .            .        (A-I-23)  
    .            .            
SR(10) = SR(X,9)   SM(10) = SM(X,9)  

DO LOOP 900 on I = 1,10 controls the columns. The columns of submatrices $\bar{X}_1$ and $\bar{X}_2$ are filled in sequence for each station counted by DO LOOP 1000. In each step in LOOP 900, four positions in submatrix $\bar{X}_1$ are filled with the power of $X$. Four positions in $\bar{X}_2$ are zeroized. The real part of the applicable integral is placed in four positions in $\bar{X}_1$ via the dummy variable SUMR. Similarly, the imaginary part is placed into $\bar{X}_2$ via SUMI. Care is taken to obtain the proper signs. Once the LHS is set for each station, four rows in the RHS matrix BB are set according to (A-I-22). The station is then augmented by the interval, XINT, and the procedure is repeated until all positions in AA and BB are filled.

For verification, the matrix prior to solution is printed out. Matrix BB is destroyed in the solution process.

The matrix is solved for the unknowns using library Subroutine DGELG. The inputs are the coefficient matrix, RHS matrix, coefficient matrix size (here set to 40 by $M=4L$), RHS matrix size, and a significance measure (here set to $1.0E-6$). The solution is returned on BB and any error indication on IER.
BB(1) to BB(10) contain the real parts of the coefficients and BB(11) to BB(20) contain the imaginary parts. The parts are combined in complex form and placed in complex array ZB. The form of ZB is

\[
\begin{align*}
ZB(1) & = \text{DELT}(0,0) = \\
ZB(2) & = \text{DELT}(0,1) = \\
& \quad \vdots \\
ZB(11) & = \text{DELT}(1,0) = (A-I-24) \\
& \quad \vdots \\
ZB(20) & = \text{DELT}(1,9) = 
\end{align*}
\]

Finally the desired coefficients are printed out on computer paper output and on cards.
MASTER ROUTINE


DL = OSWATITCH'S PARAMETER, LAMBDA
DK = STROUHAL NUMBER
RHO = SOLIDITY

TECHNIQUE INVOLVES THE SOLUTION OF 20 COMPLEX EQUATIONS AND 20 COMPLEX Unknowns. COMPLEX MANIPULATION YIELDS A MATRIX (41,40) WHICH IS SOLVED SIMULTANEOUSLY USING LIBRARY SUBROUTINE DGEQLF. THE MATRIX PRIOR TO SOLUTION IS PRINTED OUT BY ROW WITH CORRESPONDING RIGHT HAND SIDES. ERROR INDICATOR IS FROM DGEQLF AND SHOWS LOSS OF SIGNIFICANCE IF OTHER THAN 0. THE SOLUTION IS PRINTED IN COMPLEX FORMAT.

DELT(0,N), DELT(1,N), N=0,1,...,9, WHERE THE UPPER FOIL INTERFERENCE FUNCTION IS SUM ON N FROM 0 TO 9 OF DELT(1,N)*X**N, AND SIMILARLY FOR THE LOWER FOIL USING DELT(0,N).

****180 DEGREE OUT OF PHASE OPERATION****************

IMPLICIT REAL*8 (A-H,O-Z)
COMPLEX*8 ZB
DIMENSION AA(40,40), BB(40), ZB(20)
COMMON AS(10), AD(10), SR(10), SM(10)
COMMON SS, CB, SS, CCC, COC(10), COS(10)
COMMON Z, BK, BK, DL
COMMON A, RHO
COMMON/HOLD/B,C,D,E,EPS,LIM

51 FORMAT('1', 'MATRICES PRIOR TO SOLUTION')
499 FORMAT('1', 'RHS=', '10D13.7')
2000 FORMAT('1', 'ERROR INDICATION OUT=', '13')
3003 FORMAT('1', 'DELT(1,N) = ', '2E15.6', 'DELT(1,N) = ', '2E15.6')
3010 FORMAT('1', 'THE INTERFERENCE COEFFICIENTS')
3011 FORMAT('1', 'SOLUTIONS FOR THE FOLLOWING PARAMETERS')
9000 FORMAT('1', 'DL=', 'F6.3', 'DK=', 'F6.3', '10X', 'RHO=', 'F6.3', '10X', 'AL=', 'F6.3', '10X')
9001 FORMAT('1', 'SOLUTIONS FOR THE FOLLOWING PARAMETERS')
1 00 \ 20 X, DL=', 'F6.3', '10X', 'DK=', 'F6.3', '10X', 'RHO=', 'F6.3', '10X')
INITIALIZATION SECTION

SET THE CONTROLLING VARIABLES DK, DL, AND RHO

DK = 0.0 D0
DL = 0.2 D0
RHO = 0.5 D0

FOR CONVENIENCE ESTABLISH VARIOUS OTHER FIXED VARIABLES THAT ARE FUNCTIONS OF THE CONTROLLING VARIABLES. IN ADDITION VARIOUS MATRICES ARE ZEROIZED.

AL = 0.0 D0

DO 100 I = 1, 10
COC(I) = 0.0 D0
CBS(I) = 0.0 D0
AS(I) = 0.0 D0
AC(I) = 0.0 D0
PI = 3.141592653589793
IF(DK.GT.0) GO TO 1
A = 0.0 D0
GO TO 2
1 A = DK**2/(DL**2 + 4.0 D0 * DK**2)
2 B = DL * RHO**2 / 4.0 D0
C = DL * A
D = 2.0 D0 * DK * A
E = DK * RHO**2 / 2.0 D0
IF(DL.EQ.0) GO TO 3
ARG = 2.0 D0 * DK / DL
G = 3.500 * DATA2(ARG)
GO TO 4
3 G = PI / 4.0 D0
CONTINUE
4 AK = DSIN(G)
BK = DCOS(G)
Z = RHO * DSRT(DSRT(DL**2 + 4.0 D0 * DK**2)) / (2.0 D0 * DSRT(PI))

SET THE NUMBER OF STATIONS ON THE FOIL, STATION ONE, AND INTERVAL
L = 10
X = -0.9 D0
XINT = 0.2 D0

CHECK INPUTS
WRITE(6,9002)
WRITE(6,9001)DL,DK,RHO

BEGIN THE MATRIX MANIPULATION SECTION

P=0.0DO

SET UP ROW PAIRS KEYING ON X

DO 1000 J=1,L
   A=1.0DO
   JJ=2*J-1
   CALL HSUM(X,J)

SET UP ROWS KEYING ON COEFFICIENT NUMBER

DO 900 I=1,L
   AA(JJ,I)=A
   AA(JJ+1,I+10)=A
   AA(JJ+20,I+20)=A
   AA(JJ+21,I+30)=A
   AA(JJ+21,I+10)=P
   AA(JJ+20,I)=P
   SUMR=SR(I)
   SUMI=SM(I)
   AA(JJ+1,I)=SUMR
   AA(JJ+21,I+10)=SUMR
   AA(JJ+21,I+20)=SUMR
   AA(JJ+20,I+30)=SUMR
   AA(JJ,I+30)=-SUMI
   AA(JJ+1,I+20)=-SUMI
   AA(JJ+20,I+10)=-SUMI
   AA(JJ+21,I)=SMI
   A=A*X

CONTINUE

SET VALUES FOR RIGHT HAND SIDE

BB(JJ)=-SR(I)+DK*SM(2)
BB(JJ+1)=-BB(JJ)
BB(JJ+20)=-SM(I)-DK*SR(2)
BB(JJ+21)=-BB(JJ+20)
X=X+XINT
1000 CONTINUE

CHECK MATRICES
WRITE(6,51)
M=4*L
DO 520 J=1,M
WRITE(6,499)J,(AA(J,I),I=1,M),BB(J)
500 CONTINUE

SOLVE THE SIMULTANEOUS EQUATIONS
CALL DGELG(BB,AA,M,1,1.0E-6,IER)
CHECK SOLUTION
WRITE(6,2000)IER

FORM COEFFICIENTS
J=2*L
DO 600 I=1,L
600 ZB(I)=CMPLX(SNFL(BB(I)),SNFL(BB(I+20))
WRITE ZB
WRITE(6,9000)DL,DK,RHO,AL
WRITE(7,3011)DL,DK,RHO,AL
WRITE(6,3010)
DO 700 I=1,L
K=I+10
N=I-1
WRITE(6,3003)N,ZB(I),N,ZB(K)
WRITE(7,3004)ZB(I),ZB(K)
700 CONTINUE
STOP
END
A-II. SUBROUTINE WSUM

The purpose of WSUM is to provide the control for the computation of the integral

\[ I_e = \frac{R}{2} \sqrt{\frac{Q}{\pi}} \int_{-1}^{X} \frac{S^N}{(X-S)^{1/2}} \exp\left[\frac{X-S}{2Q} - \frac{R^2}{2(X-S)}\right] dS. \]  

(A-II-1)

As is demonstrated in Section III-B, (A-II-1) may be reduced to

\[ I_e = \frac{R}{2} \sqrt{\frac{Q}{\pi}} \left\{ I_{NR} - i I_{NI} \right\} \]  

(A-II-2)

where

\[ I_{NR} = \sum_{J=0}^{N+1} \frac{(-1)^J J! N! X^{N-J+1}}{(J-1)! (N-J+1)!} \int_{0}^{X+1} \left\{ U^{J-\frac{X}{2}} \exp[AU-\frac{B}{U}] \right\} \left[ \cos [2KAU+E/U] \right] \]  

(II-30)

and

\[ I_{NI} = \sum_{J=0}^{N+1} \frac{(-1)^J J! N! X^{N-J+1}}{(J-1)! (N-J+1)!} \int_{0}^{X+1} \left\{ U^{J-\frac{X}{2}} \exp[AU-\frac{B}{U}] \right\} \left[ \sin [2KAU+E/U] \right] \]  

(II-31)

where the fixed parameters A,B,C,D, and E are set in MAIN [Refer to Eqns. (A-I-11, -12, -13, -14, and -15).] and carried to WSUM in common space HOLD.
Given the integral values, the summation is a simple computer manipulation. The solution of these integrals involves three variations of the same basic numerical integration technique. The numerical integration of (III-30) and (III-31) is accomplished by a "pyramided" Simpson's rule using end correction. This technique is presented in Appendix A-III where Subroutine ZSUM is described. ZSUM prepares the numerical solution of the basic integral for whatever values of J, K, \lambda, and upper and lower limits, XO and XA, are specified.

The difficulty encountered with the numerical integration of this basic integral occurs as the result of the lower limit, XA = 0. For J < 2.5, the amplitude of the integrand may be unbounded near XA = 0. Further, except in the special case where K = 0, the SIN and COS functions in the integrand become unmanageable near zero where the controlling term in their arguments, E/U, becomes large. The resulting infinitesimal period cannot be treated by numerical techniques. For a reasonable \lambda, one may neglect the troublesome portion of the integral near zero. This is permissable since the exponential function of (-B/U) forces the amplitude to zero in the interval where the integrand is ill-behaved. When \lambda = 0, for J < 2.5, one may not neglect the integral near zero. Fortunately, in this case, one may approximate the ill-behaved portion of the integral by analytical methods.
For $\lambda = 0$, the basic integrals in Eqns. (III-30) and (III-31) become

\[
\text{SUMR} = \int_0^{X+1} U^{J-\frac{E}{2}} \cos[DU + E/U]dU \quad \text{(III-33)}
\]

and

\[
\text{SUMI} = \int_0^{X+1} U^{J-\frac{E}{2}} \sin[DU + E/U]dU \quad \text{(III-34)}
\]

Introducing the correction term of Section III-B, one may write (III-33) and (III-34) as

\[
\text{SUMR} = \text{COC} + \int AL^{J-\frac{E}{2}} U \cos[DU + E/U]dU \quad \text{(A-II-3)}
\]

and

\[
\text{SUMI} = \text{COS} + \int AL^{J-\frac{E}{2}} U \sin[DU + E/U]dU \quad \text{(A-II-4)}
\]

where the correction terms are

\[
\text{COC} = \int_0^{AL} U^{J-\frac{E}{2}} \cos[DU + E/U]dU \quad \text{(A-II-5)}
\]

and

\[
\text{COS} = \int_0^{AL} U^{J-\frac{E}{2}} \sin[DU + E/U]dU \quad \text{(A-II-6)}
\]

Integrals \text{COC} and \text{COS} are prepared by Subroutine COR.

The technique as presented in Appendix A-V provides an
approximation which reduces the correction integral to the well-known Fresnel integrals.

The limit $AL$ is selected such that (1) the numerical integration technique may be applied to the second term in (A-II-3) and (A-II-4), and (2) the correction approximation is valid over its interval.

**STEP ANALYSIS**

(Refer to the Subroutine at the end of this Appendix.)

Subroutine WSUM is called to provide the block of $L$ integrals at each station on the blades where the boundary conditions are enforced. (In this investigation $L$ is set at ten.) The subroutine inputs are $DK = K$, $DL = \lambda$, $RHO = R$, and $X$.

In the first steps several fixed parameters are established for convenience. $EPS = 1.0E-8$ is the convergence parameter used in the numerical integration process in Subroutine ZSUM. $LIM = 50$ is a step limiter also used by ZSUM [Refer to A-III]. $PI = \pi$ is used in later calculations in WSUM.

A test is made to determine which of the three cases is being dealt with. If $K = 0$, branching is executed to Step 1. If not, the next step is encountered. If $K \neq 0$ and $\lambda = 0$, the routine branches to step 2. The case $K \neq 0$ and $\lambda \neq 0$ follows at step 3.

**Step 1, $K = 0$**

The intention here is to use the numerical integration process over the entire interval. Since no SIN or COS
functions enter, one may approach closely to zero at the lower limit (XA is set to 1.0E-8). The upper limit is set to XO = X + 1. Subroutine ZSUM is called to compute the integrals \( J = 1,2,\ldots,10 \). The integral in (II-30) is returned on SUMR and the negative of the integral in (II-31) is returned on SUMI. (Note that in this case SUMI = 0.) The values are stored in the vector arrays AC and AS. The routine then jumps to step 1000 where the summation indicated in Eqns. (II-30) and (II-31) takes place.

Step 2, \( \lambda = 0, \ K \neq 0 \)

In this step the special case of \( \lambda = 0 \) is computed. The Fresnel integral is obtained, and where applicable, the correction is computed and summed with that portion of the integral obtained by numerical methods.

Observing that the value of X does not affect the correction integrals, they need be computed only once. Hence a test is made:

\[
\text{IF}(I. \text{GT}.1) \ \text{GO TO 22.}
\]

Step 22 bypasses the correction computation.

The first steps in computing the correction integrals includes the preparation of inputs for Subroutine COR and Subroutine Red. The rationale for these functions is given in A-IV and A-V. They include the following:

\[
AL = RHO/100 \quad (A-II-7)
\]

\[
AR = E/AL \quad (A-II-8)
\]
The Fresnel integrals for upper limit AR are obtained from library Subroutine CS. The explanation for CS is given on call from the library and a copy is found at the end of this Appendix. The only argument required is AR. The following is returned.

\[
\begin{align*}
CCC &= \int_0^{AR} \frac{\cos(T)}{\sqrt{2\pi T}} \, dT \quad (A-II-11a) \\
SSS &= \int_0^{AR} \frac{\sin(T)}{\sqrt{2\pi T}} \, dT. \quad (A-II-11b)
\end{align*}
\]

In preparation for their use in Subroutines COR and RED, \( (A-II-11a,b) \) are modified as follows:

\[
\begin{align*}
CCC &= \sqrt{\frac{2\pi}{E}} \left[ \frac{1}{2} - CCC \right] \quad (A-II-12a)
\end{align*}
\]

and

\[
SSS = \sqrt{\frac{2\pi}{E}} \left[ \frac{1}{2} - SSS \right]. \quad (A-II-12b)
\]

Refer to A-V for an explanation of these functions.

The various inputs in hand, Subroutine COR is called to prepare the block of \( L = 10 \) correction integrals for Eqns. \( (A-II-5, -6) \). The arguments of COR are AL, D, E, CC, SS,
ccc, and sss. the value of (a-II-5, -6) for each J is
returned on ccor and scor respectively and stored in arrays
coc and cos. coc and cos are of vector size ten and are held
in common with main for permanency. also note that (-scor)
is stored in cor for compatibility with the numerical solu-
tion of the remaining integral. the correction terms are
printed for reference.

Finally the remainder of the integrals are computed
numerically in the same manner as was used in step 1 but
with lower limit AL. The result is summed with its correction
and stored in AC and AS.

Step 3. \( K \neq 0, \lambda \neq 0 \)

In this step, one relies on the exponential function to
reduce the amplitude of the integrand near zero. That portion
of the integral from zero to 1.0E-3 is neglected. The
numerical solution is called as before and the results
stored in AC and AS.

Once the values of the integrals for \( J = 1,2,\ldots,10 \) are
in hand, the indicated summations, Eqns. (III-30, -31),
must be carried out. Step 1000 begins this operation. The
summation is carried out in a straightforward manner for
each of the \( L = 10 \) integrals. \( I_N^R \) and \( I_N^I \) are computed on
sumc and sums respectively. (Note that IFAC(M) is a function
subroutine generating the factorial of M. Refer to A-VI.)
The completed integrals are stored in arrays SR and SM and
are printed for reference.
Recalling that equation (A-II-2) contained \( Q = \lambda + 2iK \), a final complex manipulation must be made. Observe that

\[
Q = \sqrt{\lambda^2 + 4K^2} \left[ \cos(\omega) + i\sin(\omega) \right] \tag{A-II-13}
\]

where

\[
\omega = \arctan(2K/\lambda).
\]

Then

\[
\frac{R}{2}\sqrt{\frac{Q}{\pi}} = Z[\text{BK} + i\text{AK}] \tag{A-II-14}
\]

where

\[
Z = \frac{R \left[ \lambda^2 + 4K^2 \right]^{1/4}}{2\sqrt{\pi}} \tag{A-I-19}
\]

\[
\text{AK} = \sin(G) \tag{A-I-17}
\]

and

\[
\text{BK} = \cos(G) \tag{A-I-18}
\]

for

\[
G = \frac{1}{2} \arctan(2K/\lambda). \tag{A-I-16}
\]

Finally, (A-II-2) may be written

\[
I_B = Z(I_{NR \text{BK}} - I_{NI \text{AK}}) + iZ(I_{NR \text{AK}} + I_{NI \text{BK}}). \tag{A-II-15}
\]
Z, AK, and BK are computed in MAIN to eliminate repetition and are carried to WSUM in common. DO LOOP 1030 computes the real and imaginary parts of (A-II-15) for the ten integrals and returns these values to MAIN in arrays SR and SM.
SUBROUTINE WSJM(X, I)

SUBROUTINE WSUM

SUBROUTINE PROVIDES THE CONTROL FOR THE SOLUTION OF THE INTEGRAL TERM OF THE BOUNDARY EQUATIONS. INPUTS ARE DK, DL, RHO, AND X = STATION. OUTPUT IS THE EVALUATED INTEGRAL FOR THAT STATION FOR N = 0, 1, 2, ..., 9. THE COMPLEX VALUES ARE PLACED IN REAL MATRIX SR AND IMAGINARY MATRIX SM IN COMMON WITH THE MASTER PROGRAM.

THREE CASES ARE DIFFERENTIATED IN COMPUTING THE BASIC INTEGRALS FOR THE FINAL SUMMATION: DK = 0, DK = NE0, AND DL = EQ 0.


IN ALL CASES THE VALUE OF THE INTEGRALS AFTER FINAL SUMMATION IS PRINTED OUT. BEFORE THE VALUES ARE RETURNED TO MAIN A COMPLEX MANIPULATION IS CARRIED OUT. THE RESULTS OF THIS OPERATION MAY BE OBTAINED FROM THE MATRIX PRIOR TO SOLUTION.

IMPLICIT REAL*8(A-H,O-Z)
COMMON AS(10), AC(10), SR(10), SM(10)
COMMON SS, CC, SSS, CCC, COC(10), COS(10)
COMMON Z, AK, BK, DK, DL
COMMON AL, RHO
COMMON/ HOLD / B, C, O, E, EPS, LIM
COMMON/ HELD / SI(4), CI(4)
2110 FORMAT( 'O', 'O', 20X, 'DL = 0, CORRECTION USED', 'O', 20X, 'SIN AND COS FR
1ESNEL INTEGRAL VALUE FOR UPPER LIMIT AR', 'O' )
2110 FORMAT( 'O', 'O', 20X, 'DL = 0 COS AND SIN INTEGRAL CORRECTIONS', 'O' )
2119 FORMAT( 'I' )
2120 FORMAT( 'O', 'O', 20X, 'BASIC INTEGRALS PRIOR TO SUMMATION, I =', 'I2)
2130 FORMAT( 'O', 'O', 'INTEGRALS AFTER SUMMATION BUT PRIOR TO FINAL COMPLEX M
1ANIPULATION, I =', 'I2)
2300 FORMAT( 'O', 20X, 'SSS', 'D13.5', '10X', 'CCC', 'D13.5', '10X', 'AR', 'D13.5')
2311 FORMAT( ' ', 'CCCC', '10D13.5', ' ', 'COS', '10D13.5')
2301 FORMAT( ' ', 'COS INTEGRALS', ' ', '10D13.5')
2303 FORMAT( ' ', 'SIN INTEGRALS', ' ', '10D13.5')
2305 FORMAT( ' ', 'COS', '10D13.5', 'SIN', '10D13.5')
9000 FORMAT(1"'",/0",20X,'SOLUTION TO INTEGRAL(X**2.5)*SIN OR COS
1S(E/X) ON LIMITS'/",20X,'ZERO TO AL=',1PD19.3,1X,'AND FOR E=',DI
10.3/'",20X,'J=',13X,'SIN INTEGRAL',13X,'COS INTEGRAL')/

C C
EPS=1.0D-8
LIM=50
PI=3.141592653589793
IF(DK.EQ.0) GO TO 1
IF(DL.EQ.0) GO TO 2
GO TO 3
CASE OF K=0 ARGUMENT IS U**((J-5)/2)*EXP(DLAU-B/U)

C 1 XA=1.0D-8
  X0=X+1.000
  DO 10 J=1,10
  CALL ZSUM(XA,X0,J,SUMR,SUMI)
  10 AS(J)=SUMR
  GO TO 1000

C C CASE FOR K.NE.0 DL=0

C 2 IF(I.GT.1) GO TO 22

C C COMPUTE CORRECTION
AL=RHO/100.000
AR=E/AL
SS=DSIN(AR)
CC=DCOS(AR)
CALL CS(CCC,SS,AR)
WRITE(6,2100)
WRITE(6,2300)SSS,CSS,CC,AR
WRITE(6,9000)AL,E
ARG=DSQT(2.000*PI/E)
SSS=ARG**(9.500-SSS)
CCC=ARG**(0.500-CCC)
DO 20 J=1,10
  CALL COR(AL,D,E,J,CC,SS,SSS,CSS,SCOR,CCCOR)
  COC(J)=CCCOR
  COS(J)=(-SCOR)
20 CONTINUE
WRITE(6,2119)
WRITE(6,2110)
WRITE(6,2500)(COC(L),L=1,10),(COS(L),L=1,10)
COMPUTE REMAINDER OF THE INTEGRAL

\[
\begin{align*}
\text{XA} &= \text{AL} \\
\text{XD} &= \text{X} + 1.0 \\
\text{DO} 21 \text{ J} &= 1, 10 \\
\text{CALL ZSUM}(\text{XA} + \text{XD}, \text{J}, \text{SUMR}, \text{SUMI}) \\
\text{AC}(\text{J}) &= \text{CDC}(\text{J}) + \text{SUMR} \\
\text{AS}(\text{J}) &= \text{COS}(\text{J}) + \text{SUMI} \\
21 & \text{ CONTINUE} \\
\text{WRITE}(6, \text{2120}) \\
\text{WRITE}(6, \text{2505}) & \{\text{AC}(\text{L}), \text{L} = 1, 10\}, \{\text{AS}(\text{L}), \text{L} = 1, 10\} \\
\text{GO TO 1000} \\
\end{align*}
\]

CASE FOR K \text{. NE.} 0, \text{DL} \text{. NE.} 0

\[
\begin{align*}
\text{XA} &= 1.0 - 3 \\
\text{XD} &= \text{X} + 1.0 \\
\text{DO} 30 \text{ J} &= 1, 10 \\
\text{CALL ZSUM}(\text{XA} + \text{XD}, \text{J}, \text{SUMR}, \text{SUMI}) \\
\text{AC}(\text{J}) &= \text{SUMR} \\
\text{AS}(\text{J}) &= \text{SUMI} \\
30 & \text{ CONTINUE} \\
\text{GO TO 1000} \\
\end{align*}
\]

GIVEN BASIC INTEGRALS, CONVERT TO FINAL INTEGRALS

\[
\begin{align*}
\text{SUMS} &= 0.0 \\
\text{SUMC} &= 0.0 \\
\text{DO} 1020 \text{ N} &= 1, 10 \\
\text{DO} 1010 \text{ J} &= 1, N \\
\text{ST} &= (-1.0) \times (\text{J}-1) \times \text{IFAC}(N-1) \times \text{X} \times (N-J) \\
\text{ST} &= \text{ST} / (\text{IFAC}(\text{J}-1) \times \text{IFAC}(N-J)) \\
\text{SUMC} &= \text{ST} \times \text{AC}(\text{J}) \times \text{SUMC} \\
1010 & \text{ SUMS} = \text{ST} \times \text{AS}(\text{J}) \times \text{SUMS} \\
\text{SR}(\text{N}) &= \text{SUMC} \\
\text{SM}(\text{N}) &= \text{SUMS} \\
\text{SUMC} &= 0.0 \\
\text{SUMS} &= 0.0 \\
1020 & \text{ CONTINUE} \\
\text{WRITE}(6, \text{23130}) \\
\text{WRITE}(6, \text{2501}) & \{\text{SR}(\text{J}), \text{J} = 1, 10\} \\
\text{WRITE}(6, \text{2503}) & \{\text{SM}(\text{J}), \text{J} = 1, 10\} \\
\end{align*}
\]
PERFORM FINAL IMAGINARY OPERATION ON INTEGRAL VALUES

DO 1030 J=1,10
TEMP=SRT(J) SRT(J)=Z*{TEMP*AK+SM(J)*BK}
RETURN
END

1030
SUBROUTINE CS(C,S,X)

SUBROUTINE CS
PURPOSE
COMPUTES THE FRESNEL INTEGRALS.
INPUT IS STATION=X, OUTPUTS ARE THE COS INTEGRAL ON C AND SIN INTTEGRAL ON S.

Z=ABS(X)
IF(Z>4.) 1,1,2
1 C=SQRT(Z)
S=Z*C
Z=(4.-Z)*(4.+Z)
C=C*((((5.100785E-11)*Z+5.244297E-9)*Z+5.451182E-7)*Z
1+3.273309E-5)*Z+1.020418E-3)*Z+1.102544E-2)*Z+1.840965E-1)
S=S*(((((6.677681E-10)*Z+5.883318E-8)*Z+5.051141E-6)*Z
1+2.441816E-4)*Z+6.121329E-3)*Z+8.026490E-2)
RETURN
2 D=COS(Z)
S=SIN(Z)
Z=4./Z
A=(((8.769258E-4)*Z-4.169289E-3)*Z+7.970943E-3)*Z+6.792801E-3)
1+3.2095341E-4)*Z+5.972151E-3)*Z-1.606428E-5)*Z-2.493322E-2)*Z
Z=4.444091E-9
B=(((6.339262E-4)*Z+3.401469E-3)*Z+7.271690E-3)*Z+7.426246E-3)
1*Z-4.027145E-4)*Z-9.314910E-3)*Z+1.207993E-6)*Z+1.994711E-1)
Z=SQRT(Z)
C=0.5+Z*(D*A+S*B)
S=0.5+Z*(S*A-D*B)
RETURN
END
A-III. SUBROUTINE ZSUM

Subroutine ZSUM computes the value of the integral in the general Eqns. (III-30) and (III-31) by numerical methods.

\[ \text{SUMR} = \int_{\text{XA}}^{\text{XO}} \left( U^{1/2} \exp[C-U/B] \cos[D+U/E] \right) dU \quad (A-III-1) \]

\[ \text{SUMI} = \int_{\text{XA}}^{\text{XO}} \left( U^{1/2} \exp[C-U/B] \sin[D+U/E] \right) dU \quad (A-III-2) \]

The technique utilized is Simpson's rule with end correction. Noting the behavior of the integrand, especially near \( XA = 0 \), it is not satisfactory to compute the integral using a fixed interval. Instead a "pyramided" scheme is used which automatically reduces the size of the subintervals until the scheme can compute the integral to within the limits of a satisfactory convergence criterion.

The general form of Simpson's rule with end correction is

\[ \text{SUM} = \frac{H}{15} \left( 7Y_1 + 16Y_2 + 14Y_3 + 16Y_4 + \ldots + 16Y_{N-1} + 7Y_N \right) \]

\[ + \frac{H^2}{15} (Y_1' - Y_N') \]

or

\[ \text{SUM} = \frac{H}{15} \left( 7(Y_1 + Y_N) + 16 \sum_{i=1}^{N/2} Y_{2i} + 14 \sum_{i=1}^{N/2} Y_{2i+1} \right) + \frac{H^2}{15} (Y_1' - Y_N') \]

(A-III-3)
where \( Y_1 \) and \( Y_1' \) are the value of the integrand and the derivative of the integrand evaluated at station (i), and \( H \) is the length of the subinterval [Refer to Fig. 8].

![Integration Diagram](image)

**Figure 8. INTEGRATION DIAGRAM**

Figure 9 demonstrates the "pyramid" algorithm. \( X_1 \) and \( X_N \) are the first and last station of the interval over which \( Y(X) \) is to be integrated, i.e., \( X_1 = XA \) and \( X_N = X0 \). \( N \) is the number of stations at which the integrand must be evaluated. There are \((N-1)\) equal sub-intervals of length \( H \) in \( \text{INT} = X_N - X_1 \).

<table>
<thead>
<tr>
<th>Step</th>
<th>( N )</th>
<th>( H )</th>
<th>( X_1 )</th>
<th>coefficients of ( X_1 )</th>
<th>coefficients of ( X_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>INT/2</td>
<td>7</td>
<td>16</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>INT/4</td>
<td>7</td>
<td>16  14</td>
<td>16  7</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>INT/8</td>
<td>7</td>
<td>16  14  16</td>
<td>16  14  16  7</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>INT/16</td>
<td>7</td>
<td>16  14  16  14</td>
<td>16  14  16  16</td>
</tr>
</tbody>
</table>

**PYRAMID SCHEME**

**Figure 9**

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The accuracy of Simpson's approximation increases with decreased H. The pyramided technique progressively computes the integral, doubling the number of sub-intervals each time, until the value of the computation ceases to change to within some desired criterion, EPS.

Figure 9 shows the change in N and H with each step. The coefficients for each step are shown in their relative positions according to station. Observe that once a $Y_i$ is evaluated for a given step, it is used in each succeeding step and need not be reevaluated. Further note that the first time a $Y_i$ is used, its coefficient is 16; thereafter it is 14.

The integrand in (A-III-1) is

$$Y_R(U) = U^{J-\frac{5}{2}} \exp(CU-B/U)\cos(DU+E/U), \quad (A-III-4)$$

and it may be shown that

$$Y_R'(U) = U^{J-\frac{7}{2}} \exp(CU-B/U) \left\{ \cos(DU+E/U)(J-5/2+CU+B/U) 
- (DU-E/U)\sin(DU+E/U) \right\}. \quad (A-III-5)$$

The integrand in (A-III-2) is

$$Y_I(U) = U^{J-\frac{5}{2}} \exp(CU-B/U)\sin(DU+E/U) \quad (A-III-6)$$

and
\[ Y_1'(U) = U^{J-\frac{3}{2}} \exp(\frac{C(U-B)}{U}) \{ \sin(\frac{D+3}{U})(J-5/2+C(U-B)/U) \]
\[ + \frac{(D-U)\cos(\frac{D+3}{U})}{U} \} \]  

(A-III-7)

STEP ANALYSIS

(Refer to the subroutine at the end of this section.)

Subroutine ZSUM arguments are as follows:
- \( XA \) = Lower limit
- \( XO \) = Upper limit
- \( J \) = Summation index set in WSUM. [Refer to Eqns. (II-30, -31).]

The following parameters are brought to ZSUM via common space HOLD: B, C, D, E, EPS, and LIM. B, C, D, and E are computed in MAIN [Refer to Eqns. (A-I-12, -13, -14, and -15).]. EPS and LIM are set in WSUM. XA, XO, and J are also set in WSUM. The values of (A-III-1, -2) are returned on SUMR and SUMI.

Equations (A-III-4, -5, -6, and -7) are first evaluated at the left endpoint. AA = \( C(U-B)/U \) is the argument of the exponential function. If AA is less than (-172), it is assumed that the exponential term drives the integrand and its derivative to zero by branching to statement 6 (Computer underflow errors are generated for smaller values of the exponent.). BB = \( D(U+3)/U \), the argument of the SIN and COS functions, and CC = \( D-U \), a term used in \( Y_1' \), are computed. The evaluations of \( Y_R(XA) \) and \( Y_I(XA) \) are obtained on YAR and YAI from the arithmetic statement functions YR and YI.
where \( Y_R \) is (A-III-4) and \( Y_I \) is (A-III-6). EE and FF are additional terms used in the derivative. Finally the values of \( Y_R'(X) \) and \( Y_I'(X) \) are computed on YARP and YAIP.

Statement 6 sets all values \( Y_R(X) \), \( Y_I(X) \), \( Y_R'(X) \), and \( Y_I'(X) \) to zero for small values of the exponential function (those cases where \( AA \leq -172 \)).

The same process is executed for the right end point. Here \( Y_R(XO) = YOR \), \( Y_I(XO) = YOI \), \( Y_R'(XO) = YORP \), and \( Y_I'(XO) = YOIP \).

The first step in computing the internal points is to set the interval of integration in \( DINT = XO - XA \). Next, various summation parameters are zeroed including SOR and SOI, the sums of the \( Y_{RI} \)'s and \( Y_{II} \)'s which will be multiplied by the coefficient 14 [Refer to Eqn. (A-III-3)] and SER and SEI, those sums to be multiplied by 16. SP sets the number of subintervals in each step (initially set at 2) and K sets the number of terms in SER and SEI to be computed in each step (initially set at 1).

DO LOOP 100 controls the "pyramid" level or step from one to LIM. \( H \) is the length of each subinterval in each step, \( H = DINT/SP \).

On each step, the \( Y_I \)'s computed the previous step for coefficient 16 (summed in SER and SEI) are added to those with coefficient 14 (SOR and SOI). SER and SEI are then zeroed in preparation for the next computation. XE is the dummy variable in this portion of the calculation. The initial \( Y_I \) is computed at \( XE = XA + H \).
DO LOOP 10 computes the K new $Y_i$'s for SER and SEI. AA and BB are computed as before. $Y_R(XE)$ and $Y_I(XE)$ are computed on YER and YEI and summed on SER and SEI respectively. Finally the dummy variable is augmented by 2H and the loop continues.

SUMR and SUMI hold the values of (A-III-1) and (A-III-2) for any given step. TEMPR and TEMPI are used as temporary storage for the values in SUMR and SUMI computed in the previous step. Once a new set of values are computed in SUMR and SUMI, they are compared with the previous values. If both do not differ from the previous values by more than EPS, convergence criteria is met and the loop terminates. Should the process reach LIM iterations without convergence, a warning is printed and the existing values of the integrals are accepted.
SUBROUTINE ZSUM(XA, X0, J, SUMR, SUMI)

-------------------------------
SUBROUTINE ZSUM

PROGRAM FOR THE COMPUTATION OF THE INTEGRAL

METHOD UTILIZED IS A PYRAMIDED SIMPSON'S RULE WITH END CORRECTION.
LIMITS ARE FROM XA TO X0. PYRAMID LIMIT IS LIM. CONVERGENCE IS
MEASURED BY EPS. B, C, D, AND E ARE FIXED FUNCTIONS OF DK, DL, AND RHO.
TWO INTEGRALS ARE COMPUTED, THE REAL AND IMAGINARY PORTION OF THE
INTEGRAL IN THE BOUNDARY EQUATIONS. THE VALUES ARE RETURNED ON SUMR
AND SUMI.

A WARNING IS GENERATED IN THE CASE OF NON CONVERGENCE.

-------------------------------
IMPLICIT REAL*8(A-H, O-Z)
COMMON/HOLD/B, C, D, E, EPS, LIM
YR(U, J, AA, BB) = U**(-1)*EXP(AA)*DCOS(BB)/(U*DSQRT(U))
YI(U, J, AA, BB) = U**(-1)*EXP(AA)*(-DSIN(BB))/(U*DSQRT(U))

2 FORMAT('1', 'WARNING: ZSUM NON CONVERGENT')

ESTABLISH LEFT END POINT

U=XA
AA=C*U-B/U
IF(AA .LE. (-172.000)) GO TO 6
BB=D*U+E/U
CC=D*U-E/U
YAR=YR(U, J, AA, BB)
YAI=YI(U, J, AA, BB)
EE=EXP(AA)*U**(-1)/(U*DSQRT(U))
FF=DFLOAT(J)-2.500+C*U+B/U
YARP=EE*(DCOS(BB)*FF-CC)*DSIN(BB))
YAIP=(-EE)*(DSIN(BB)*FF+CC)*DCOS(BB))
GO TO 3
6 YAR=0.000
YAI=0.000
YARP=0.000
YAIP=0.000
3 CONTINUE
ESTABLISH THE RIGHT END POINT

U=X0
AA=C*U-B/U
BB=D*U+E/U
CC=D*U-E/U
YOR=YR(U,J,AA,BB)
YOL=YL(U,J,AA,BB)
EE=DEXP(AA)*U**(J-2)/(U**DSORT(U))
FF=DFLOAT(J)-2.500+C*U+B/U
YORP=EE*(DCOS(BB)+FF-CC*DSIN(BB))
YOLP=(-EE)*(DSIN(BB)+FF+CC*DCOS(BB))

ESTABLISH INTERVAL

DINT=XC-XA
SOR=J.0DO
SOI=0.0DO
SER=0.0DO
SEI=0.0DO
SUMR=0.0DO
SUMI=0.0DO

INITIAL SPACING AND NUMBER OF TERMS IN EVEN SUM

SP=2.0DO
K=1

LOOP SPECIFYING INTERVAL

DO 100 M=1,LIM
H=DINT/SP
SET OLD EVEN SUM INTO ODD SUM
SOR=SOR+SER
SOI=SOI+SEI
XE=XA+H
ZEROIZE EVEN SUM
SR=0.0DO
SEI=0.0DO

LOOP TO ESTABLISH EVEN SUM

DO 10 I=1,K
AA=C*XE-B/XE
IF(AA.LE.(-172.0DO)) GO TO 4
BB=D*XE+E/XE
YOR=YR(XE,J,AA,BB)
YOL=YL(XE,J,AA,BB)
GO TO 5
4 YER=0.0D0
  YE=0.0D0
5 CONTINUE
  SER=SER+YER
  SEI=SEI+YEI
10 XE=EX+2.0D0*H

FORM THE INTEGRAL
TEMPR=SUMR
TEMPI=SUMI
SUMR=H*(7.0D0*(YAR+YOR)+16.0D0*SER+14.0D0*SOR)/15.0D0
SUMR=SUMR+H**2*(YARP-YORP)/15.0D0
SUMI=H*(7.0D0*(YA+YO)+16.0D0*SEI+14.0D0*SOI)/15.0D0
SUMI=SUMI+H**2*(YAPIP-YOIP)/15.0D0
L=M

TEST FOR CONVERGENCE

TTTR=DABS(TEMPR-SUMR)
TTTI=DABS(TEMPI-SUMI)
IF(TTTR.LE.EPS.AND.TTTI.LE.EPS) GO TO 101

PROVIDE FOR ERROR STATEMENT IF NO CONVERGENCE

IF(M.EQ.LIM) GO TO 1
GO TO 9
1 WRITE(6,2)
9 K=K+2
SP=SP*2.0D0
100 CONTINUE
101 CONTINUE
RETURN
END
A-IV. SUBROUTINE COR

Subroutine COR controls the computation of the correction integrals (A-II-5) and (A-II-6) for the case $\lambda = 0$. The integrals are

$$CCOR = \int_{0}^{AL} U^{J-\frac{5}{2}} \cos(DU+E/U) dU \quad (A-IV-1)$$

and

$$SCOR = \int_{0}^{AL} U^{J-\frac{5}{2}} \sin(DU+E/U) dU \quad (A-IV-2)$$

where $D$ and $E$ are specified in (A-I-14,-15). An analytic approximation for these integrals exists in theory; however, computer numerical error prohibits reliable use of this technique for $J \geq 5$ and $R > 2$ [Refer to the Discussion of Results for an explanation]. For this reason, the approximation is used only for $J \leq 4$. This range includes those troublesome integrals having integrands with infinite amplitude near the origin. Standard numerical integration is used for $J > 4$ where the amplitude of the integrands becomes small near zero thus enabling one to approach zero sufficiently close to ensure that the omitted interval is negligible.

CASE FOR $J \leq 4$

Under the assumption that, for a properly selected $AL$, as $U$ approaches zero the term $E/U$ dominates $DU$ in the
arguments of SIN and COS then equations (III-40) and (III-41) are reasonable approximations of (A-IV-1) and (A-IV-2) even when a reduced number of terms is included. Such a selection of AL is AL = R/100 which ensures (E/U)/DU \geq 1.0E^4 for U \leq AL.

Subroutine COR carries out the operation of (III-40) and (III-41) where

\[
CCOR = \int_{0}^{AL} u^{j - \frac{5}{2}} \cos(\frac{E}{U}) dU - D \int_{0}^{AL} u^{j - \frac{3}{2}} \sin(\frac{E}{U}) dU
\]

\[
-\frac{D^2}{2} \int_{0}^{AL} u^{j - \frac{1}{2}} \cos(\frac{E}{U}) dU + \frac{D^3}{6} \int_{0}^{AL} u^{j + \frac{1}{2}} \sin(\frac{E}{U}) dU
\]

\[
(A-IV-3)
\]

and

\[
SCOR = \int_{0}^{AL} u^{j - \frac{5}{2}} \sin(\frac{E}{U}) dU + D \int_{0}^{AL} u^{j - \frac{3}{2}} \cos(\frac{E}{U}) dU
\]

\[
-\frac{D^2}{2} \int_{0}^{AL} u^{j - \frac{1}{2}} \sin(\frac{E}{U}) dU - \frac{D^3}{6} \int_{0}^{AL} u^{j + \frac{1}{2}} \cos(\frac{E}{U}) dU.
\]

\[
(A-IV-4)
\]

Due to the inaccuracy in the computation of the integrals by Subroutine RED, a power of U greater than 1.5 is avoided by reducing the number of terms in the truncated series by one each time the series is used. That is, for J = 1, four terms are used; for J = 2, three terms are used; and so forth until only one term is used for J = 4.
CASE FOR J > 4

For this case Subroutine ZSUM is used to numerically compute the integral over the interval XA to AL where XA is picked as close to the origin as computer time and accuracy warrant (XA = 1.0E-4 was selected for these results). It should be noted, however, that in some cases the value of the integral over this interval may be less than the convergence limit, EPS, used in ZSUM in which case the results may suffer from inaccuracy. Provided the error is on the order of EPS (EPS should be the upper limit) such inaccuracies are not significant and should not affect the end result.

STEP ANALYSIS

(Refer to the subroutine at the end of this Appendix.)

The inputs to Subroutine COR include:

AL = Upper limit of the correction integral
J = Summation index in equation (III-30) and (III-31)

\[ D = \frac{2K^3}{\lambda^2 + 4K^2} \]

\[ E = \frac{KR^2}{2} \]

C = COS(E/AL)
S = SIN(E/AL)

\[ SS = \int_{0}^{\ AL} \frac{1}{U^{3/2}} \ SIN(E/U)dU \]

\[ CC = \int_{0}^{\ AL} \frac{1}{U^{3/2}} \ COS(E/U)dU \]
D and E are fixed parameters computed in MAIN; C, S, SS, and CC are formed in Subroutine WSUM. CCOR and SCOR return the values of (A-IV-3) and (A-IV-4) to WSUM.

The first step is to determine if \( J = 1, 2 < J < 4, \) or \( J > 4. \) For the case \( J = 1, \) four integrals are computed in (A-IV-3,-4) of the form

\[
CINT = \int_0^{AL} U^{M-\frac{5}{2}} \cos(E/U) dU \tag{A-IV-5}
\]

and

\[
SINT = \int_0^{AL} U^{M-\frac{5}{2}} \sin(E/U) dU \tag{A-IV-6}
\]

where \( M = J, J+1, J+2, \) and \( J+3. \) These integrals are easily computed in four step DO LOOP 10 utilizing Subroutine RED. RED reduces (A-IV-5,-6) to the general form

\[
\text{Integral} = f(AL,N,E) + \int_0^{AL} \frac{1}{U^{\frac{3}{2}}} \sin(E/U) dU. \tag{A-IV-7}
\]

The values of these integrals are stored in two arrays, SI and CI, of dimension four. Note that should a full four term truncated series be desired for \( J > 1, \) only one additional integral is needed in (A-IV-3,-4) each time. That is, in terms of the \( J \) of the preceding calculation, values of CINT and SINT are required with \( M = J+1, J+2, J+3, \) and \( J+4. \) Since the first three are already on hand, DO LOOP 13 sets SI(I+1) into SI(I) and CI(I+1) into CI(I). The
fourth position is filled with the newly calculated values of CINT and SINT for $M = J+4$. However, assuming error in RED for $J > 4$, it was elected to truncate the series by one additional term for each case, $J = 2, 3, \text{ and } 4$. Hence, zero is set into $SI(4)$ and $CI(4)$.

One assumes that this reduction in accuracy is still acceptable compared to the error introduced by an attempt to use numerical integration.

In the case $J > 4$, a branch is provided to statement 21 where subroutine ZSUM is used to calculate the integral between AL and XA. Although this procedure simplifies the handling of COR results by WSUM, inaccuracies of the order of EPS are expected as previously mentioned. One might expect greater accuracy using another technique which did not compute the integral piecemeal.
SUBROUTINE COR(AL, D, E, J, C, S, SS, CC, SCOR, CCOR)

SUBROUTINE COR
COMPUTATION OF THE CORRECTION TO INTEGRAL U**(J-2.5)*COS OR
SIN(DU+E/U) FOR THE CASE OF DL=0 AND J=1 THROUGH 5. NUMERICAL
INTEGRATION IS USED FOR CASES J=6 THROUGH TEN

THE CORRECTION IS FOR THE ILLBEHAVED PORTION OF THE INTEGRAL NEAR
ZERO WHICH CANNOT BE COMPUTED BY NUMERICAL METHODS. LIMITS ON THE
CORRECTION ARE ZERO TO AL. D AND E ARE FIXED FUNCTIONS OF DK AND
RHO. J IS A FUNCTION OF N. S AND C ARE THE FUNCTIONS SIN(E/AL)
AND COS(E/AL). SS AND CC ARE FUNCTIONS OF THE BASIC FRESNEL
INTEGRALS.

THE METHOD UTILIZES A TRUNCATED SIN AND COS SERIES APPROXIMATION
AND A REDUCTION SUBROUTINE WHICH REDUCES THE INTEGRAL TO A BASIC
FUNCTION OF THE FRESNEL INTEGRAL.

THE VALUES OF THE SIN AND COS INTEGRAL CORRECTION ARE RETURNED ON
SCOR AND CCOR RESPECTIVELY

PRINTED OUTPUT INCLUDES THE UPPER LIMIT OF THE CORRECTION, THE
FUNCTION J, AND THE VALUES OF THE BASIC CORRECTION INTEGRALS
BEFORE TREATMENT IN THE TRUNCATED SERIES. THE FINAL CORRECTION
AFTER THE SERIES OPERATION IS PRINTED OUT BY SUBROUTINE WSUM.

IMPLICIT REAL*8(A-H, O-Z)
COMMON/HELD/SI(4), CI(4)

TEST FOR CASES J GREATER THAN 6, GREATER THAN TWO OR EQUAL TO ONE
IF(J.GE.5) GO TO 21
IF(J.GE.2) GO TO 12

CASE J EQUAL JNE
UTILIZE THE REDUCTION FORMULA TO OBTAIN THE BASIC CORRECTION TERMS.

K=J
DO 10 I=1, 4
  CALL RED(AL, E, K, S, C, SS, CC, SI(I), CI(I))
SI(I)=SI(I)+S
CI(I)=CI(I)+C

10 CONTINUE
10 K=K+1
20 GO TO 14
12 DO 13 I=1,3
13 SI(I)=SI(I+1)
13 CI(I)=CI(I+1)
C C C C C C
CASES J BETWEEN THREE AND FIVE SET INTEGRALS TO ZERO FOR THOSE CASES WHERE COMPUTER NUMERICAL ERROR IS EXPECTED
14 SI(4)=0.0D0
14 CI(4)=0.0D0
14 CONTINUE
14 UTILIZE THE SIN AND COS TRUNCATED SERIES TO OBTAIN THE FINAL INTEGRAL CORRECTION.
14 SCOR=SI(1)+D*(CI(2)-D*(SI(3)/2.0D0+D*CI(4)/6.0D0))
14 CCOR=CI(1)-D*(SI(2)+D*(CI(3)/2.0D0-D*SI(4)/6.0D0))
14 GO TO 30
C C C C C C
PROVIDE INTEGRAL CORRECTION BY NUMERICAL INTEGRATION TECHNIQUES FOR THOSE REGIONS WHERE ANALYTICAL SOLUTION IS NOT POSSIBLE DUE COMPUTER NUMERICAL ERROR
21 XA=1.0D-4
21 XO=AL
21 CALL ZSUM(XA,XO,J,CCOR,SCOR)
21 SCOR=-SCOR
30 CONTINUE
30 RETURN
30 END
A-V. SUBROUTINE RED

Subroutine RED provides the value of the general integrals

$$ SINT = \int_{0}^{\frac{A}{J}} U^{-\frac{1}{2}} \sin(E/U) dU \quad (A-V-1) $$

and

$$ CINT = \int_{0}^{\frac{A}{J}} U^{-\frac{1}{2}} \cos(E/U) dU \quad (A-V-2) $$

by reducing them to functions of the form

$$ SINT = F_1(A,J,E) + \int_{0}^{\frac{A}{J}} \frac{1}{U^{\frac{3}{2}}} \sin(E/U) dU \quad (A-V-3) $$

and

$$ CINT = F_2(A,J,E) + \int_{0}^{\frac{A}{J}} \frac{1}{U^{\frac{3}{2}}} \cos(E/U) dU \quad (A-V-4) $$

The integrals in (A-V-3,-4) are functions of the Fresnel integrals as follows:

$$ CC = \int_{0}^{\frac{A}{J}} \frac{1}{U^{\frac{3}{2}}} \cos(E/U) dU = \sqrt{\frac{2\pi}{E}} \left[ \frac{1}{2} - C(E/A) \right] \quad (A-V-5) $$

and

$$ SS = \int_{0}^{\frac{A}{J}} \frac{1}{U^{\frac{3}{2}}} \sin(E/U) dU = \sqrt{\frac{2\pi}{E}} \left[ \frac{1}{2} - S(E/A) \right] \quad (A-V-6) $$

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where

\[ C(E/A) = \int_{0}^{E/A} \frac{1}{\sqrt{2\pi T}} \cos(T) dT \]  \hspace{1cm} (A-V-7)

and

\[ S(E/A) = \int_{0}^{E/A} \frac{1}{\sqrt{2\pi T}} \sin(T) dT. \] \hspace{1cm} (A-V-8)

That is, \( S(t) \) and \( C(t) \) for any \( J \) may be reduced by integration by parts to functions of the basic Fresnel integrals.

Integration by parts provides the following relations for (A-V-1).

If \( J = 1 \),

\[ C(t) = \int_{0}^{A} \frac{1}{U^{3/2}} \cos(E/U) dU. \] \hspace{1cm} (A-V-9)

For \( J = 2 \),

\[ C(t) = 2[A^{1/2} \cos(E/A) - E \int_{0}^{A} \frac{1}{U^{3/2}} \sin(E/U) dU]. \] \hspace{1cm} (A-V-10)

And for \( J \geq 3 \),
\[
C_{\text{INT}} = \frac{(J-2)!}{(2J-3)!} \left\{ A^{\frac{1}{2}} \cdot \right. \\
\left. \left[ \sum_{N=1}^{\left[ \frac{J}{2} \right]} \left( \frac{2}{J-2N-1} \right)^{N-1} \frac{(A-2N)^{J-2N}}{(J-2N)!} \right] + \frac{\text{INT}}{2^{2J-3}E^{J-1}} \right\} 
\]

where

\[
\text{INT} = 2^{2J-3}E^{J-1} (-1)^{\left( \frac{J}{2} \right)} \int_0^{\frac{A}{U}} \frac{1}{U^{\frac{3}{2}}} \cos(E/U) \, dU, \quad (-1)^J < 0
\]

or

\[
\text{INT} = 2^{2J-3}E^{J-1} (-1)^{\left( \frac{J}{2} \right)} \int_0^{\frac{A}{U}} \frac{1}{U^{\frac{3}{2}}} \sin(E/U) \, dU, \quad (-1)^J > 0.
\]

Similarly, it may be shown that for (A-V-2) one obtains:

For \( J = 1 \),

\[
S_{\text{INT}} = \int_0^{\frac{A}{U^{\frac{3}{2}}}} \sin(E/U) \, dU,
\]

if \( J = 2 \),

\[
S_{\text{INT}} = 2[A \sin(E/A) + E \int_0^{\frac{A}{U^{\frac{3}{2}}}} \cos(E/U) \, dU],
\]
and for $J \geq 3$,

$$\text{SINT} = \frac{(J-2)!}{(2J-3)!} \left\{ A^{\frac{J}{2}} \right. \right.$$ 

$$\left. \cdot \left[ \sin\left(\frac{E}{A}\right) \sum_{N=1}^{J+1} \left[ \frac{2(J-2N)!}{(J-2N)!} \frac{(-1)^{N-1}2^{N-1}J-2N}{2(N-1)} \right] \right. \right.$$ 

$$\left. + \cos\left(\frac{E}{A}\right) \sum_{N=1}^{J} \left[ \frac{2(J-2N-1)!}{(J-2N-1)!} \frac{(-1)^{N-1}2^{N-1}J-2N-1}{2(N-1)} \right] \right. \right.$$ 

$$\left. + \text{INT} \right\} \quad (A-V-15)$$

where

$$\text{INT} = 2^{2J-3}E^{J-1}(-1)^{\frac{J}{2}} \int_{0}^{\frac{A}{\sqrt{2}}} \sin\left(\frac{E}{U}\right) dU, \quad (-1)^{J} < 0 \quad (A-V-16a)$$

or

$$\text{INT} = 2^{2J-3}E^{J-1}(-1)^{\frac{J}{2}-1} \int_{0}^{\frac{A}{\sqrt{2}}} \cos\left(\frac{E}{U}\right) dU, \quad (-1)^{J} > 0. \quad (A-V-16b)$$

The notation $[\text{ARG}]$ indicates the sign of ARG with the largest integer less than the absolute value of ARG. The use of $[\text{ARG} \pm 0.1]$ ensures the proper value is selected and excludes the possibility of computer round off error.

**STEP ANALYSIS**

(Refer to the subroutine at the end of this Appendix.)

The arguments of Subroutine RED include:

$A = \text{Upper limit of the correction integral.}$
\[ E = \frac{K^2 R^2}{2} \]

\[ J = \text{Related to the summation index in (III-30,-31)} \]

\[ S = \sin\left(\frac{E}{\Lambda}\right) \]

\[ C = \cos\left(\frac{E}{\Lambda}\right) \]

\[ SS = \int_{0}^{A} \frac{1}{U^{\frac{3}{2}}} \sin\left(\frac{E}{U}\right) dU \]

\[ CC = \int_{0}^{A} \frac{1}{U^{\frac{3}{2}}} \cos\left(\frac{E}{U}\right) dU \]

\( E \) is set in MAIN and carried in common to WSUM and as an argument through COR to RED. A, S, C, SS, and CC are formed in Subroutine WSUM and are carried as arguments to RED via COR. \( J \) is set by COR. The values of \((A-V-1,-2)\) are returned to Subroutine COR on SINT and CINT.

A test is made to determine if \( J < 3 \). If it is, the routine branches to statement 1000.

If \( J \geq 3 \), equations \((A-V-11)\) and \((A-V-15)\) apply. \( I_1 \) and \( I_2 \) are the summation limits where \( \text{INT(ARG)} = [\text{ARG}] \) described previously. The primary series DO LOOP computes the first summation in \((A-V-11, -15)\). The secondary series DO LOOP computes the second summation. A test is made to check the criteria of \((A-V-12,-16)\) and proper branching is initiated. Statement 31 computes \((A-V-12a,-16a)\) for \((-1)^J \leq 0\), and statement 32 computes \((A-V-12b,-16b)\) for \((-1)^J > 0\). Finally, a statement 40 combines the terms to form \((A-V-11,-15)\) on SINT and CINT.
For the case $J < 3$, a test is made to further restrict the case to $J = 2$ or $J = 1$. For $J = 2$, $(A-V-10, -14)$ are used. For $J = 1$, $(A-V-9, -13)$ are computed.
SUBROUTINE RED(A,E,J,S,C,SS,CC,SINT,CINT)

SUBROUTINE RED

COMPUTE INTEGRAL X**(J-2.5)SIN OR COS(E/X) BY REDUCTION TO FRESNEL INTEGRAL

METHOD UTILIZES A REDUCTION FORMULA DEVELOPED FROM INTEGRATION BY PARTS. LIMITS ARE ZERO TO A. E IS A FIXED FUNCTION ON DK AND RHO. S AND C ARE SIN AND COS FUNCTIONS OF E AND A. SS AND CC ARE FUNCTIONS OF THE BASIC FRESNEL INTEGALS. THE VALUES OF THE COMPUTATION ARE RETURNED ON SINT AND CINT FOR THE SIN AND COS INTEGRAL RESPECTIVELY.

IMPLICIT REAL*8(A-H,O-Z)

IF(J.LT.3) GO TO 1000

CASE OF J.GT.2

J1=INT(FLOAT(J)/2.0+0.1)
J2=INT(FLOAT(J)/2.0-0.1)
SUMER=0.0DO
SUMS=0.0DO

PRIMARY SERIES DO LOOP

DO 10 N=1,J1
KK=J-2*N
TERM=DFLOAT(IFAC(2*KK))/DFLOAT(IFAC(KK))
TERM=TERM*(-1.0DO)***(N-1)*2.0DO***(4*N-3)
TERM=TERM*A**KK*E**(2*(N-1))
SUMER=SUMER+TERM

10

SECONDARY SERIES DO LOOP

DO 20 N=1,J2
KK=J-2*N-1
TERM=DFLOAT(IFAC(2*KK))/DFLOAT(IFAC(KK))
TERM=TERM*(-1.0DO)***(N-1)*2.0DO***(4*N-1)

20

END
TERM = TERM ** K ** E ** (2 ** N - 1)

20 SUMS = SUMS + TERM

FORM INTEGRAL PART OF SUM

TEST FOR ODD OR EVEN J

KK = (-1) ** J
IF( KK) 31, 32, 32

ODD J

31 KK = INT( FLOAT(J) / 2.0)
SUB = 2.0D ** (2 * J - 3) * E ** (J - 1) * (-1.0D) ** KK
SINT = SUB ** SS
CINT = SUB ** CC
GO TO 40

EVEN J

32 SUB = 2.0D ** (2 * J - 3) * E ** (J - 1)
SINT = SUB ** (-1.0D) ** (J / 2 - 1) ** CC
CINT = SUB ** (-1.0D) ** (J / 2) ** SS

FORM THE CORRECTION SUMMATION

40 SUB = DFLOAT(IFAC(J - 2)) / DFLOAT(IFAC(2 * J - 3))
B = DSQRT(A)
SINT = SUB ** (B ** (S ** SUMP + C ** SUMS) + SINT)
CINT = SUB ** (B ** (C ** SUMP - S ** SUMS) + CINT)
GO TO 3000

1000 IF(J.LT.2) GO TO 2000

CASE OF J = 2

Z = DSQRT(A)
SINT = 2.0D ** (Z ** S + E ** CC)
CINT = 2.0D ** (Z ** C - E ** SS)
GO TO 3000

CASE OF J = 1

2000 SINT = SS
CINT = CC
3000 CONTINUE

WRITE(6,9000) J, SINT, CINT
9000 FORMAT('0', 20X, i3, 10X, 1PD15.4, 10X, D15.4)
RETURN
END
A-VI. **SUBROUTINE IFAC**

Function subroutine IFAC returns the value of the argument factorialed, e.g., IFAC(5) = 5·4·3·2·1. It has provisions for IFAC(0) = 1. A copy of the subroutine follows.
FUNCTION IFAC(I)

SUBROUTINE IFAC
IFAC IS A FUNCTION SUBROUTINE DESIGNED TO PRODUCE FACTORIALS

IF(I.EQ.0) GO TO 11
IF(I.EQ.1) GO TO 11
IFAC=1
II=I-1
DO 10 J=1,II
10 IFAC=IFAC*(I-J)
GO TO 12
11 IFAC=1
12 RETURN
END
APPENDIX B

STEADY STATE POTENTIAL SOLUTION

B-I. COLLOCATION TECHNIQUE

In order to investigate the perturbation velocity potential along a cascade blade for the steady state and \( \pi \) phase shift in adjacent blades, one may use the following:

\[
\psi_{K=0}(X,0) = \frac{1}{\sqrt{\pi \lambda}} \left\{ \int_{-1}^{X} \frac{1}{\sqrt{X-S}} \, dS + \int_{-1}^{X} \frac{1}{\sqrt{X-S}} \exp \left[-\frac{\lambda R^2}{4(X-S)} \right] \, dS \right\} \\
+ \frac{1}{\sqrt{\pi \lambda}} \sum_{N=0}^{L=1} \left\{ \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \exp \left[-\frac{\lambda R^2}{4(X-S)} \right] \, dS \right\} \\
- \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \, dS \right\} \quad (I-I-47)
\]

The second and third integrals are of the form

\[
I = \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \exp \left[-B/(X-S) \right] \, dS, \quad \text{where} \quad B = \lambda R^2/4. \quad (B-I-1)
\]

Letting

\[
U = B/(X-S), \quad (B-I-2)
\]

one may write (B-I-1) as

\[
I = \sqrt{B} \int_{B/X+1}^{\infty} \frac{1}{U^{3/2}} (X-B/U)^N \, e^{-U} \, dU \quad (B-I-3)
\]
or

\[ I = \sqrt{B} X^N \int_0^{\frac{1}{B/XU}} e^{-U} \frac{1}{u^{1/2}} \, dU \]  
(B-I-4)

The binomial expansion provides

\[(1-B/XU)^N = \sum_{K=0}^{N} \frac{(-1)^K N!}{(N-K)!K!} (B/XU)^K. \]  
(B-I-5)

Hence (B-I-4) becomes

\[ I = \sqrt{B} X^N \sum_{K=0}^{N} \frac{(-1)^K N!}{(N-K)!K!} (B/XU)^K \int_0^{\frac{1}{B/XU}} u^{-(K+1)/2} e^{-U} \, dU. \]  
(B-I-6)

It may be shown that integration by parts provides

\[ \int_{\frac{1}{B/XU}}^{\infty} e^{-x} \, dx = \left[ -\frac{1}{M-1} \frac{e^{-x}}{x^{m-1}} \right]_{\frac{1}{B/XU}}^{\infty} \]

\[ - \frac{1}{M-1} \int_{\frac{1}{B/XU}}^{\infty} \frac{e^{-x}}{x^{m-1}} \, dx, \quad M > 0 \]  
(B-I-7)

where \( M \) may be any half odd integer greater than zero, e.g., \( M = 1/2, 3/2, 5/2, \) etc. Continuation of the integration process yields a general relationship

\[ \int_{\frac{1}{B/XU}}^{\infty} \frac{e^{-x}}{x^{m}} \, dx = \left[ e^{-p^2} \sum_{I=1}^{m-1} \frac{m-1}{I!} \frac{(M-I-1)!(-1)^{I-1}}{(M-I)!} \right] \]

\[ + \frac{(-1)^{m-1}}{M-1!} \sqrt{\pi} \text{ERFC}(p) \]  
(B-I-8)
where

$$\text{ERFC}(F) = \frac{1}{\sqrt{\pi}} \int_{F}^{\infty} e^{-U} \frac{dU}{U^{3/2}} = \frac{1}{\sqrt{\pi}} [1 - \text{erf}(F)]. \quad (B-I-9)$$

For the case \(N = 0\), \((B-I-6)\) is

$$I = \sqrt{B} \int_{B/(X+1)}^{\infty} \frac{e^{-U}}{U^{3/2}} dU. \quad (B-I-10)$$

The last integral in (III-47) may be written

$$\int_{-1}^{X} \frac{S^{N}}{\sqrt{X-S}} dS = 2\sqrt{X+1} \sum_{K=0}^{N-K} (-1)^{K} \frac{N! (X+1)^{K}}{(2K+1)K! (N-K)!} \quad (B-I-11)$$

and observe that for the case \(N = 0\), \((B-I-11)\) becomes

$$\int_{-1}^{X} \frac{1}{\sqrt{X-S}} dS = 2\sqrt{X+1}. \quad (B-I-12)$$

Combining Eqns. \((B-I-6,-10,-11, \text{and -12})\) in (III-47) and replacing \(N\) with \(J-1\) and \(K\) with \(K-1\), one arrives at the final computer compatible equation

$$\psi_{K=0}(X,0) = \frac{1}{\sqrt{\pi \lambda}} \left[2\sqrt{X+1} + 2\sqrt{B} \sum_{-\frac{3}{2}}^{b/(X+1)} \right]$$

$$- \sum_{J=1}^{L} \sum_{K=1}^{(J-1)!} \frac{(-1)^{K-1} (J-1)!(X-J-K)}{(K-1)!(J-K)!} \left[ \varphi_{0,J-1} \frac{(X+1)^{K-1} \sqrt{X+1}}{2K-1} \right]$$

$$- \varphi_{1,J-1} B^{K-1} \sqrt{B} \sum_{-(K+\frac{1}{2})}^{b/(X+1)} \right] \] \quad (B-I-13)$$

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where

$$SSUM(A, FSQ) = \frac{1}{2} \int_{FSQ}^{\infty} x^A \exp(-x) dx.$$  \hspace{1cm} (B-I-14)

SSUM is computed in Subroutine SSUM and will be dealt with in Appendix B-I-B.

B-I-A. THE MAIN PROGRAM

The purpose of MAIN is the computation of (B-I-13). Inputs to the program are the controlling parameters, blade spacing, $R$, and thickness parameter, $\lambda$, as well as the values of the interference coefficients obtained from the collocation coefficient solution of Appendix A.

Step Analysis

(Refer to the routine at the end of Appendix B-I.)

Double precision is used in Subroutine SSUM and single precision is used in MAIN. Double precision arguments to and from SSUM are carried on variables beginning with $W$.

The inputs $DLAM = \lambda$, $DK = K$, and $RHO = R$, and the coefficients of the upper and lower interference functions ($\varphi_{lm}$ and $\varphi_{om}$, $m = 1, 2, \ldots, 10$) are read into the routine from data cards and printed out immediately for verification. (Note that $K$, the reduced frequency, is used for accounting purposes on the output only.) Array DEL(20) carries the coefficients with the lower blade coefficients in the first ten positions.
The inputs positioned, various fixed parameters are formed and the station, \(X\), is initialized.

\[
\begin{align*}
B & = \lambda R^2 / 4 \\
BH & = \sqrt{B} \\
C & = 1 / \sqrt{\pi \lambda} \\
D & = -3/2 \\
X & = -0.9
\end{align*}
\]

DO LOOP 100 is the station control loop which generates a potential value at each of 20 positions on the blade \((X = -0.9, -0.8, \ldots, 1.0)\).

In preparation for the use of Subroutine SSUM, the lower limit is formed,

\[
WFSQ = B/(X+1).
\]

SSUM is called to form

\[
SSUM[D,B/(X+1)] = \frac{1}{2} \int_B^\infty \frac{e^{-u}}{u^{3/2}} du.
\]

With \(AA = \sqrt{X+1}\), the first two terms in (B-I-13) are formed:

\[
BB = 2\sqrt{X+1} + 2B \cdot SSUM[-3/2,B/(X+1)].
\]

DO LOOP 200 accomplishes the \(J\) summation in (B-I-13) while LOOP 300 carries out the \(K\) summation. (Recall that
L = 10.) Each term in the summation is built up in a straightforward manner. Subroutine SSUM is required to compute

$$SSUM(A, WFSQ) = \frac{1}{2} WFSQ \int_0^\infty U^A e^{-U} dU,$$

where

$$A = -(K+\frac{1}{2}).$$

Function subroutine IFAC is also used. This subroutine produces the factorials of its argument and is explained in Appendix A-VI.

The summation loop completed, the summation term is combined with BB and C to form $POT = \psi_{K=0}(X,0)$ of equation (B-I-13). Finally POT is printed together with its station.

B-I-B. SUBROUTINE SSUM

The purpose of SSUM is to produce the value of the general integral

$$\frac{1}{2} \int_{-\infty}^{\infty} X^A \exp(-X) dX, \quad A = \pm 1/2, \pm 3/2, \pm 5/2, \ldots \quad (B-I-15)$$

Only the derivation of the procedure for $A < 0$ will be presented in detail here. The case for $A > 0$ is derived in a similar manner. This latter case is not of interest in the potential solution.
The online function subroutine, INT(A), generates the sign of A with the largest integer less than A, e.g., INT(-3/2) = -1. The subroutine, ERFC(F), produces the compliment of the error function using $F^2$ as the lower limit. It should be understood that $(N/2)! = \frac{N}{2} \cdot \frac{N-1}{2} \cdot \frac{N-2}{2} \ldots \frac{1}{2}$ and that $(-1/2)! = 1$.

Now for demonstration purposes, (B-I-8) is expanded for the cases $A = -3/2$ and $A = -5/2$.

\[
\frac{1}{2} \int_{F^2}^{\infty} \frac{e^{-x}}{x^{5/2}} \, dx = \frac{1}{2} \left[ \frac{2}{eF^2} \sqrt{\pi} \text{ERFC}(F) \right] \quad \text{(B-I-16)}
\]

\[
A = -5/2
\]

\[
\frac{1}{2} \int_{F^2}^{\infty} \frac{e^{-x}}{x^{5/2}} \, dx = \frac{1}{2} \left[ \frac{2}{eF^2} \left( \frac{2}{3F^3} - \frac{2^2}{3.1F} \right) + \frac{2^2 \sqrt{\pi} \text{ERFC}(F)}{3.1} \right] \quad \text{(B-I-17)}
\]

A routine may be generated in the following manner. Observe that if one defines $I = \text{INT}(A)$, there are $I$ terms in the series (Those terms not including the error function term.). The numerator is increased by two in each succeeding term. The denominator of the first term includes $(2I-1)$; each succeeding denominator in the series is multiplied in turn by $(2I-1)-2$, $(2I-1)-4$, .... The power of $F$ in the denominator begins with $(2I-1)$ and is decreased by two in each succeeding term. The sign changes with each term. Also
note that the numerical coefficient of the error function term is the same as that for the last series term with the sign reversed.

Step Analysis

(Refer to the subroutine at the end of Appendix B-I.)

The inputs to the subroutine are \( A \) and \( FSQ = F^2 \). The value of (B-I-15) is returned on \( SUM \). In the first steps various fixed parameters are formed, the summation parameter is zeroed, and \( INT(A) \) is determined. A test is made to determine if \( A > 0 \) or \( A \leq 0 \). Here only the branch \( A \leq 0 \) is of interest although the structure of the unused portion of the subroutine for \( A > 0 \) is similar.

For \( A \leq 0 \), statement 10 tests for the special case \( A = \frac{1}{2} \) (or \( I = 0 \)). This case cannot be generated by the algorithm used for other cases. If \( A = -\frac{1}{2} \), statements 12 and 14 set

\[
SUM = \frac{\sqrt{\pi}}{2} \text{ERFC}(F)
\]

and the routine returns to MAIN.

If \( A < -\frac{1}{2} \), statement 13 begins the algorithm. After several preliminary steps, \( \text{DO LOOP 11} \) forms the series pointed out previously. The method takes advantage of the observations made in the introductory material. The series formed on \( SUM \), it is divided by the exponential and the error function term is added.

The result is returned to MAIN.
PROGRAM FOR THE COMPUTATION OF THE STEADY STATE POTENTIAL 
ALONG THE BLADE SURFACE. INPUTS ARE THE REAL COEFFICIENTS 
OF THE INTERFERENCE FUNCTION AND THE SET OF PARAMETERS 
DL, DK, AND RHO. OUTPUT IS THE NUMERICAL VALUE OF THE POTENTIAL 
AT EACH STATION ALONG THE UPPER SURFACE OF THE BLADE.

**********************************************************************

IMPLICIT REAL*8 (A)
REAL*8 DBLE
DIMENSION DEL(20)
5 FORMAT('0', 20X, 'THE INTERFERENCE COEFFICIENTS'//)
10 FORMAT(E20.7, 20X, E20.7)
15 FORMAT('1', 20X, 'DEL(0', I1, ')=', 1PE14.7, 10X, 'DEL(1', I1, ')=', E14.7)
30 FORMAT(3D20.3)
31 FORMAT(16'0', 20X, 'VALUE OF THE POTENTIAL ALONG THE BLADE; 0', 20X, 'STATION', 13X, 'POTENTIAL')
32 FORMAT(16'1', 20X, 'POTENTIAL SOLUTION FOR THE FOLLOWING PARA-
1ETERS; 0', 20X, 'DL=', F7.3, 10X, 'DK=', F7.3, 10X, 'RHO=', F7.3)
400 FORMAT('0', 20X, 'X=', F5.2, 10X, 'POT=', 1PE14.7)
1000 FORMAT('1')

INPUT THE VARIABLES AND COEFFICIENTS

READ(5, 30) DLAM, DK, RHO
DO 20 I=1, 10
K=I+10
READ(5, 10) DEL(I), DEL(K)
CONTINUE
WRITE(6, 32) DLAM, DK, RHO
WRITE(6, 5)
DO 16 I=1, 10
K=I-1
M=I+10
WRITE(6, 15) K, DEL(I), K, DEL(M)
CONTINUE
WRITE(6, 31)
COMMENCE THE COMPUTATION BY ESTABLISHING VARIOUS FIXED PARAMETERS.

B = DLAM*RHO**2/4.0
BH = SORT(B)
PI = 3.141593
C = 1.0/SORT(PI*DLAM)
D = -3.0/2.0
X = -0.9
XINT = 0.1

STATION CONTROL LOOP

DO 100 I = 1, 20
WFSQ = DBLE(B/(X+1.0))
CALL SSUM(D, WFSQ, WSUM)
AA = SORT(X+1.0)
SUM = SNGL(WSUM)
BB = 2.0*AA+BH*2.0*SUM
TTERM = 0.0

J LOOP

DO 200 J = 1, 10

K LOOP

DO 300 K = 1, J
ISIG = J-1
IR = K-1
IX = ISIG-IR
CC = FLOAT(IFAC(ISIG))/FLOAT(IFAC(IR))*FLOAT(IFAC(ISIG-IR))
CC = CC*(-1.0)**IR*X**IX*2.0
DD = (X+1.0)**IR*AA/(2.0*FLOAT(IR)+1.0)
A = FLOAT(IR)+0
CALL SSUM(A, WFSQ, WSUM)
SUM = SNGL(WSUM)
EE = BB**IR*BH**SUM
TERM = CC*(DEL(J)*DD-DEL(J+1)*EE)
TTERM = TTERM+TERM

300 CONTINUE
200 CONTINUE
POT = C#(BB-TTERM)
WRITE(6, 400) X, POT
100 X = X+XINT
WRITE(6, 1000)
STOP
END
SUBROUTINE SSUM(A, FSQ, SUM)

******************************************************************************************************************

SUBROUTINE SSUM

PRODUCE THE SOLUTION OF THE INTEGRAL 0.5*W**A*EXP(-W) ON THE LIMITS FSQ=PI#DLAM/X+1 TO INFINITY.

******************************************************************************************************************

IMPLICIT REAL*4(A-C), REAL*8(D-H, O-V)
SUM=0.0D0

OBTAIN SIGNIFICANT INTEGER POWER OF W WITH SIGN

I=INT(A)
F=DSORT(FSQ)
V=DSORT(3.141592653589793)
DEN=DEXP(FSQ)

TEST SIGN OF A
IF(A)10,19,20
'A' NEGATIVE

10 IF(I.EQ.0) GO TO 12
GO TO 13
12 TER1=-1.0D0
GO TO 14
13 DD=1.0D0
EE=2.0D0
I=-I
I2=2*I-1
DD=11 K=1, I
I4=I2-2*(K-1)
DD=DD*DFLOAT(I4)
TER1=DFLOAT((-1)**(K+1))*EE/DD
TER=TER1/(F**I4)
EE=EE*2.0D0
SUM=SUM+TER
CONTINUE
15 SUM=SUM/DEN
14 SUM=(SUM-TER1*V*DERFC(F))/2.0D0
GO TO 30
"A" POSITIVE

20 IF (1.EQ.0) GO TO 22
   GO TO 23
22 TER2 = 0.5DO
   SUM = F/DEN
   GO TO 24
23 DD = 1.0DO
   EE = 1.0DO
   TER2 = 1.0DO
   I = I + 1
   I2 = 2 * I + 1
   DO 21 K = 1, I
      I3 = I2 - 2 * K
      TER1 = E * I3
      TER = TER1 * TER2
      DD = DD * DFLOAT (I3)
      EE = EE * 2.0DO
      TER2 = DD / EE
      SUM = SUM + TER
   CONTINUE
21 SUM = SUM / DEN
24 SUM = (SUM + TER2 * V*DERFC(F)) / 2.0DO
30 RETURN
END
B-II. SONIC WIND TUNNEL WALL INTERFERENCE TECHNIQUE

Development of the Sandeman transonic wind tunnel wall interference solution for the steady case resulted in the following perturbation potential function from Section III-E:

$$\psi(X,0) = -\frac{1}{\sqrt{\pi \lambda}} \left\{ \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \, dS \right\}$$

$$+ 2 \sum_{N=1}^{\infty} \left\{ \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \exp\left[-\lambda \frac{N^2 d^2}{X-S}\right] \, dS \right\}$$

(III-60)

To compare this solution with the cascade problem, (d) is related to blade spacing $R$ by $d = R/2$ [Refer to Fig. 7]. Further, as shown in Eqn. (III-18),

$$v(S) = \psi_y(X, Y = 0) = -1, \text{ for } K = 0.$$  

It follows that the potential may be written:

$$\psi(X,0) = \frac{1}{\sqrt{\pi \lambda}} \left\{ \int_{-1}^{X} \frac{1}{\sqrt{X-S}} \, dS \right\}$$

$$+ 2 \sum_{N=1}^{\infty} \left\{ \int_{-1}^{X} \frac{1}{\sqrt{X-S}} \exp\left[-\lambda \frac{N^2 R^2}{X-S}\right] \, dS \right\}$$

(B-II-1)

Integrating the first term and making the proper substitution in the second term one may write

$$\psi(X,0) = \frac{1}{\sqrt{\pi \lambda}} \left\{ 2 \sqrt{X+1} + 2 \sum_{N=1}^{\infty} \sqrt{\frac{\lambda N^2 R^2}{4}} \int_{\frac{2N^2 R^2}{4(X+1)}}^{\infty} e^{-U} \, dU \right\}$$

(B-II-2)
The basic integral within the summation may be expressed in terms of the error function, i.e.,

\[ \int_{\frac{p^2}{U}}^{\infty} \frac{e^{-U}}{U^{3/2}} \, dU = \frac{2 e^{-\frac{p^2}{F}}}{F} - 2 \sqrt{\pi} \text{ERFC}(F) \]  \hspace{1cm} (B-II-3)

where

\[ p^2 = \frac{\lambda N^2 R^2}{4(X+1)} \]  \hspace{1cm} (B-II-4)

and

\[ \text{ERFC}(F) = 1 - \text{erf}(F) \]  \hspace{1cm} (B-II-5)

Hence, the potential function:

\[ \psi(X,0) = \frac{2}{\sqrt{\pi} \lambda} \left\{ \sqrt{\frac{X}{X+1}} + \sum_{N=1}^{\infty} \left(2\sqrt{\frac{X+1}{X+1}} e^{-\frac{\lambda N^2 R^2}{4(X+1)}} \right) \right\} - \sqrt{\pi \lambda N^2 R^2} \text{ERFC}\left[ \sqrt{\frac{\lambda N^2 R^2}{4(X+1)}} \right] \]  \hspace{1cm} (B-II-6)

Computation of this function is easily carried out by computer. The series is truncated when the inclusion of additional terms only changes the sum by a value less than the convergence criterion, EPS.

**Step Analysis**

(Refer to the routine at the end of this appendix.)

The inputs to the routine are DLAM = \( \lambda \) and RH0 = \( R \).

DK = \( K \) is inputed for accounting purposes in the output only.

The output includes the above parameters and the value of
the perturbation velocity potential at each of twenty stations along the blade.

The control parameters and various fixed parameters are calculated prior to entering the control loops. DO LOOP 100 provides control to calculate the potential at each of the stations beginning at \( X = -0.9 \) and increasing by increments of \( \Delta X = 0.2 \). DO LOOP 50 computes the series summing terms until convergence is reached. Again, the function subroutine ERFC is a standard on line library routine performing the operation in (B-II-5).

Once the series is computed, statement 51 forms the potential on POT which is subsequently printed along with the applicable station location.
SOLUTION OF THE POTENTIAL ALONG THE BLADE BY THE WALL INTERFERENCE TECHNIQUE

10 FORMAT('1//',//,,//','.20X,'VALUE OF THE POTENTIAL ALONG THE BLADE
10-WALL INTERFERENCE TECHNIQUE:',//,'0',.20X,'SOLUTION FOR THE FOLLOWING PARAMETERS:',//,'0',.20X,'DL=','F7.3, 10X, 'DK=','F7.3, 10X, 'RHO=','F7.3/,/'10',.20X,'STATION',.13X,'POTENTIAL'

52 FORMAT('0',.20X,'X=','F5.2,10X,'PCT=','1PE15.5)

SET THE CONTROL PARAMETERS

DLAM=0.2
DK=0.0
RHO=1.0
WRITE(6,10)DLAM,DK,RHO

PI=3.141593
C=SQRT(PI*DLAM)
X=-0.9

STATION CONTROL LOOP

DO 100 I=1,20
SUM=0.0
A=X+1.0
AH=SQRT(A)

SERIES CONTROL LOOP

DO 50 N=1,500
B=DLAM*N*N*RHO*RHO/(4.0*A)
BH=SQRT(B)
D=2.0*A*AH/EXP(B)
TERM=D-N*RHO*C*ERFC(BH)
TEMP=SUM
SUM=SUM+TERM
TEST FOR CONVERGENCE

IF(ABS(SUM-TEMP).LE.1.0E-5) GO TO 51

50 CONTINUE
51 POT=2.0*(AH+SM)/C
WRITE(6,52)X,POT
X=X+0.1
100 CONTINUE
WRITE(6,1000)
STOP
END
LIST OF REFERENCES


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This thesis presents an analysis of transonic potential flow through an oscillating unstaggered thin plate cascade. A collocation technique is used involving the superposition of adjacent blade "isolated foil" potentials with interference potentials of unknown strength. Imposition of flow tangency requirements leads to integral equations for the unknown source distributions of the interference potentials. Results presented include the interference coefficients for the steady and oscillating cases.

The steady case is extended to the determination of the potentials along the blade and compares favorably with a parallel solution using a Laplace transform approach to the sonic wind tunnel wall interference problem.
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A theoretical analysis of unsteady transonic cascade flow.
A theoretical analysis of unsteady trans