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Singular Perturbation of the Wave Equation

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Although Dirichlet conditions for the wave equation is a classical case of a non-well-posed problem in partial differential equations, there has been recent interest in studying those conditions under which the problem is well-posed. This question is of applied interest (V. Barcilon, Mathematika 15 (1968), 93) as well as theoretical interest (A. I. Abdul-Latif and J. B. Diaz, Appl. Anal. 1 (1971), 1). In this paper the onset of nonuniqueness is studied by considering a higher order equation with a small parameter ε in the limit $\varepsilon \to 0$. The method is quite similar to the method of small viscosity used to study the onset of shock waves.

A classical "non-well-posed" problem is the wave equation with Dirichlet conditions. However in the past 30 years there has been revived interest in conditions under which the problem is well posed [1–8]. The original paper [1] of Bourgin and Duffin was concerned with the wave equation in a rectangle $0 \leqq x \leqq S, 0 \leqq t \leqq T$, and the basic theorem was

THEOREM 1 *Let $u(x, t)$ be a C^2 solution of the wave equation $u_{xx} - u_{tt} = 0$ a rectangle $0 \leqq x \leqq S$, $0 \leqq t \leqq T$, with zero boundary data. Then $u \equiv 0$ iff S/T is irrational.*

This result has been extended by many people with essentially the same hypotheses and essentially the same conclusion. John [2] analyzed non-rectangular domains and showed that they can be mapped uniquely into a rectangle with either rational or irrational sides. Extensions have been made to Neumann conditions and conditions of the third kind [1, 2, 6, 7]. Extensions have been made to higher dimensions and the *E-P-D* equation

[5, 6, 8], and to ultra-hyperbolic equations [9] with obvious modifications to the irrationality condition. The results are the same—if the ratio of the sides meets a certain condition, then there exists an infinite number of eigenfunctions to the homogeneous problem.

All of these papers are concerned with the uniqueness of the homogeneous solution and are not concerned with either the construction of a solution for non-homogeneous conditions or with the stability of their results under perturbation. The author has recently considered problems of the singular perturbation of fourth order elliptic equations reducing to second order elliptic equations [10, 11] of the form

$$\mathscr{E}^2\mathscr{L}_4u\pm\mathscr{L}_2u = 0, \tag{1}$$

constructing asymptotic approximations for explicit solutions for given boundary data on a rectangle. In the case where the coefficient of $\mathscr{L}_2$ is negative the solution is given by the smooth solution of $\mathscr{L}_2u = 0$ with the lowest order boundary data, plus a correction of boundary layer character, of order $\mathscr{E}^2$ in magnitude [10]. In the case where the coefficient of $\mathscr{L}_2$ is positive, the solution of (1) is given by the smooth solution of $\mathscr{L}_2u = 0$ with the lowest order boundary data, plus a correction which consisted of high frequency oscillations of order $\mathscr{E}^2$.

Thus it seems appropriate to study the explicit solutions of the model problem

$$\mathscr{E}^2(\partial^4u/\partial x^4+\partial^4u/\partial t^4)+2(\partial^2u/\partial x^2)-2(\partial^2u/\partial t^2) = 0 \tag{2}$$

with the rectangle $0 \leqq x \leqq S$, $0 \leqq t \leqq T$ with the boundary conditions on the boundary $\delta\Gamma$

$$u(\delta\Gamma) = f \tag{3}$$

$$\partial^2u/\partial n^2\,(\delta\Gamma) = g \tag{4}$$

where $\partial^2u/\partial n^2$ is the second normal derivative and f and g are given functions. A singular perturbation approach is not meaningful because the $\mathscr{E} = 0$ case of (2) with (3) is not clearly defined, in contrast to (1). Instead we have chosen a model problem which can be solved explicitly in terms of Fourier series. The degeneration of the series solution as $\mathscr{E} \to 0$ should then shed some light on the desired problem.

It is clearly necessary and sufficient to consider two cases of boundary conditions

Case 1

$$u(\delta\Gamma) = f = \begin{cases} f(S, t) & x = S \\ 0 & \text{otherwise} \end{cases} \tag{3a}$$

$$(\partial^2 u/\partial n^2) = \begin{cases} (\partial^2 u/\partial x^2) \\ 0 \end{cases} = g(S, t) \qquad \begin{matrix} x = S \\ \text{otherwise} \end{matrix} \tag{4a}$$

Case 2

$$u(\delta\Gamma) = f = \begin{cases} f(x, T) & t = T \\ 0 & \text{otherwise} \end{cases} \tag{3b}$$

$$(\partial^2 u/\partial n^2) = \begin{cases} (\partial^2 u/\partial t^2) \\ 0 \end{cases} = g(x, T) \qquad \begin{matrix} t = T \\ \text{otherwise} \end{matrix} \tag{4b}$$

We consider first Case 2, and separate variables, with the hypothesis that f and g have convergent Fourier sine series expansions. Writing $U(x, t) = X(x)R(t)$ we obtain, since the conditions in the t direction are homogeneous,

$$R(t) = R_n(t) = \sin n\pi t/T \tag{5}$$

$$\begin{aligned} \mathrm{X}(x) = {} & A_n \sin\surd[1+\surd(1-\mathscr{E}^2\lambda^2)]x/\mathscr{E} \\ & +B_n \sin\surd[1-\surd(1-\mathscr{E}^2\lambda_n)]x/\mathscr{E} \end{aligned} \tag{6}$$

where

$$\lambda_n = [\surd(2\pi)n/T]\surd[1+(n^2\pi^2\mathscr{E}^2/2T^2)], \quad n = 1, 2, \ldots. \tag{7}$$

From the form (6) it is clear that there is a cut off at $\lambda_n = \mathscr{E}^{-1}$, where the arguments become complex and then (6) should be replaced by

$$\begin{aligned} \mathrm{X}(x) = {} & A_n \sin \alpha\, nx/\mathscr{E} \cosh \delta nx/\mathscr{E} \\ & +B_n \cos \alpha\, nx/\mathscr{E} \sinh \delta nx/\mathscr{E} \end{aligned} \tag{8}$$

where

$$\begin{aligned} \alpha_n^2 &= (\mathscr{E}\lambda_n+1)/2 \\ \delta_n^2 &= (\mathscr{E}\lambda_n-1)/2 \end{aligned} \tag{9}$$

Presuming that we can solve for A_n and B_n, which we will try to do shortly, it is clear that for low values of λ_n the x behavior is characterized by a high frequency ($\sim \surd(2)\, x/\mathscr{E}$) term and a low frequency ($\sim n\pi\, x/T$) term, while for very large values of n both x terms are characterized by oscillatory, strong exponentials. The changeover is at $\lambda_n = \mathscr{E}^{-1}$ or

$$(n\pi\mathscr{E}/T)^2 = \surd(2)-1. \tag{10}$$

(The special case of (6) or (8) when $\lambda\mathscr{E}_n = 1$ is a trivial matter and we do not consider it any further.)

Is it possible to solve for A_n and B_n? Let f_n and g_n be the coefficients of the Fourier sine expansions of $f(t)$ and $g(t)$ respectively. Then, for $\mathscr{E}\lambda_n < 1$ we have

$$A_n = \frac{f_n(1-\sqrt{[1-\mathscr{E}^2\lambda_n^2]})+\mathscr{E}^2 g_n}{2\sqrt{[1-\mathscr{E}^2\lambda_n^2]}\,\sin\sqrt{[1+\sqrt{(1-\mathscr{E}^2\lambda_n^2)}]}\,S/\mathscr{E}} \tag{11}$$

$$B_n = \frac{f_n(1+\sqrt{1-\mathscr{E}^2\lambda_n^2})+\mathscr{E}^2 g_n}{2\sqrt{[1-\mathscr{E}^2\lambda_n^2]}\,\sin\sqrt{[1-\sqrt{(1-\mathscr{E}^2\lambda_n^2)}]}\,S/\mathscr{E}} \tag{12}$$

From an inspection of (11) and (12) we may conclude that

LEMMA 1. For $\mathscr{E}\lambda_n < 1$ we may uniquely solve for A_n and B_n in (6) provided

$$\sqrt{[1\pm\sqrt{(1-\mathscr{E}^2\lambda_n^2)}]}\,S/\mathscr{E} \neq p\pi \tag{13}$$

for some integer $p > 0$.

Assuming (13) holds then for small values of $\mathscr{E}$ and/or n we consider (11) and (12). For A_n (11) implies that both the coefficient of f_n and that of g_n are of order $\mathscr{E}^2$ and the denominator is a highly oscillatory function of $\mathscr{E}$. For B_n (12) implies that the coefficient of f_n is of order 1 and that of g_n is of order $\mathscr{E}^2$, while the denominator is slowly oscillating. Thus

$$u(x, t) \approx \sum f_n \sin n\pi t/T \sin n\pi x/S + O(\mathscr{E}^2) \tag{14}$$

The terms $O(\mathscr{E}^2)$ consist of two highly oscillatory terms involving both f_n and g_n plus a slow correction involving g_n.

For $\mathscr{E}\lambda_n > 1$ it is an elementary exercise to see that it is always possible to determine A_n and B_n. It is easy to show that

$$u(x, t) \approx \sum f_n \sin n\pi t/T \frac{\sinh \beta x/\mathscr{E} \cos \alpha x/\mathscr{E}}{\sinh \beta S/\mathscr{E} \cos \alpha S/\mathscr{E}} + O(\mathscr{E}^2) \tag{15}$$

so that the high frequency portion of the solution essentially vanishes except near the right hand boundary.

What is the nature of the restriction (13), which is the analogue of the irrationality criterion? Using the definition (7) for λ_n^2, it is easy to reduce (13) to the problem—is there a solution in square integers (n^2, p^2) of

$$n^2/T^2\,(1+[n^2\pi^2\mathscr{E}^2/2T^2]) = p^2/S^2\,(1-[p^2\pi^2\mathscr{E}^2/2S^2])\,? \tag{16}$$

Since the left side has the character of a parabola opening upwards and the right side has the character of a parabola opening downward, it is clear that

LEMMA 2 *There is at most one pair of integers n and p, other than* (0, 0), *such that* (13) *fails to hold, if* $\mathscr{E} > 0$.

Thus the infinite set of eigenvalues of the degenerate problem ($\mathscr{E} = 0$) become at most one eigenvalue.

We turn now to the case where the non-zero data is given by (3b) and (4b), parallel to the x axis. Then we obtain

$$\begin{aligned} X(x) &= X_n(x) = \sin n\pi x/S \\ R(t) &= R_n(t) = C_n \sinh \sqrt{[\sqrt{(k_n+1)}+1]}t/\mathscr{E} \\ &\qquad + D_n \sin \sqrt{[\sqrt{(k_n+1)}-1]}t/\mathscr{E} \end{aligned} \tag{17}$$

where

$$k_n = (2\mathscr{E}^2 n^2\pi^2/S^2)-(n^4\pi^4\mathscr{E}^4/S^4) \tag{18}$$

It is clear that there are two cutoff values of n, one where k_n becomes negative and one where k_n becomes less than -1, and that the form of (17) is indicative of the case for small n and/or $\mathscr{E}$. We claim that we have

LEMMA 3 *For* $0 < n\pi\mathscr{E}/S < 2$ *Eq.* (17) *describes the t dependence, while for* $2 < n\pi\mathscr{E}/S < 1+\sqrt{2}$ *we have*

$$\begin{aligned} R_n(t) &= C_n \sinh \sqrt{[\sqrt{(1+k_n)}+1]}t/\mathscr{E} \\ &\qquad + D_n \sinh \sqrt{[1-\sqrt{(1+k_n)}]}t/\mathscr{E} \end{aligned}$$

and for $1+\sqrt{2} < n\pi\mathscr{E}/S$ *the t dependence is exactly as in* (8), *with x and T replaced by t and S respectively.*

We again ask for the solvability of the boundary value problem. For small n, below the first cutoff we have

$$C_n = \frac{f_n\sqrt{(k_n+1)}+\mathscr{E}^2 g_n}{2\sqrt{(1+k_n)}\sinh\sqrt{[\sqrt{(1+k_n)}+1]}T/\mathscr{E}} \tag{19}$$

$$D_n = \frac{f_n\sqrt{(k_n+1)}-\mathscr{E}^2 g_n}{2\sqrt{(1+k_n)}\sin\sqrt{[\sqrt{(1+k_n)}-1]}T/\mathscr{E}} \tag{20}$$

Thus we have

LEMMA 4 *The coefficient C_n of the hyperbolic term in* (17) *can always be found while the coefficient D_n of the trigonometric term in* (17) *can be found if*

$$\sqrt{[\sqrt{(1+k_n)}-1]}T/\mathscr{E} \neq p\pi \tag{21}$$

for some integer $p > 0$.

An elementary calculation shows that (21) is equivalent to the criterion (13) with n and p reversed, so we have the same conclusion as before about the overall solvability of the boundary value problem. However, now it is only D_n about which there can be any question.

We may again conclude that for small $\mathscr{E}$ the numerator of (19) is of order $\mathscr{E}^2$, while the first part of D_n is of order 1 and the second of order $\mathscr{E}^2$. Thus we see that

$$u(x, t) \approx \sum f_n \sin n\pi x/S \sin n\pi t/S + O(\mathscr{E}^2) \tag{22}$$

where the terms of $O(\mathscr{E}^2)$ consist of a boundary layer term due to C_n and a slowly oscillatory correction from D_n. As expected the boundary layer occurs near $t = T$, where the data is given. Above the cutoff frequency we have behavior similar to the previous case.

We summarize our results in the following

THEOREM *Equation* (2) *with boundary data* (3) *and* (4) *is uniquely solvable except for at most a single eigenvalue* (*given by* (13)). *The solution is characterized, below a certain pair of cutoff frequencies, by the solution of the degenerate problem* ($\mathscr{E} = 0$) *plus a boundary correction in the t direction and high frequency oscillations in the x direction, both of order* $\mathscr{E}^2$.

The different behavior in the two directions is to be expected from the work of [10] and [11]. It is interesting that the infinite number of eigenvalues of the degenerate case have almost all disappeared. Even the one which remains is quite sensitive to the values of S, T and $\mathscr{E}$. This gives some justification to those who have ignored the solvability question in doing applied problems [12].

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